

# Adaptive thresholding estimation of a Poisson intensity with infinite support

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**Abstract** The purpose of this paper is to estimate the intensity of a Poisson process  $N$  by using thresholding rules. In this paper, the intensity, defined as the derivative of the mean measure of  $N$  with respect to  $ndx$  where  $n$  is a fixed parameter, is assumed to be non-compactly supported. The estimator  $\tilde{f}_{n,\gamma}$  based on random thresholds is proved to achieve the same performance as the oracle estimator up to a logarithmic term. Oracle inequalities allow to derive the maxiset of  $\tilde{f}_{n,\gamma}$ . Then, minimax properties of  $\tilde{f}_{n,\gamma}$  are established. We first prove that the rate of this estimator on Besov spaces  $\mathcal{B}_{p,q}^\alpha$  when  $p \leq 2$  is  $(\log(n)/n)^{\alpha/(1+2\alpha)}$ . This result has two consequences. First, it establishes that the minimax rate of Besov spaces  $\mathcal{B}_{p,q}^\alpha$  with  $p \leq 2$  when non compactly supported functions are considered is the same as for compactly supported functions up to a logarithmic term. This result is new. Furthermore,  $\tilde{f}_{n,\gamma}$  is adaptive minimax up to a logarithmic term. When  $p > 2$ , the situation changes dramatically and the rate of  $\tilde{f}_{n,\gamma}$  on Besov spaces  $\mathcal{B}_{p,q}^\alpha$  is worse than  $(\log(n)/n)^{\alpha/(1+2\alpha)}$ . Finally, the random threshold depends on a parameter  $\gamma$  that has to be suitably chosen in practice. Some theoretical results provide upper and lower bounds of  $\gamma$  to obtain satisfying oracle inequalities. Simulations reinforce these results.

**Keywords** Adaptive estimation, Model selection, Oracle inequalities, Poisson process, Thresholding rule

**Mathematics Subject Classification (2000)** 62G05 62G20

## 1 Introduction

### 1.1 Motivations

Statistical inference for the problem of estimating the intensity of some Poisson process is considered in this paper. For this purpose, we assume that we are given observations of a Poisson process on  $\mathbb{R}$  and our goal is to provide a data-driven procedure with good performance for estimating the intensity of this process.

This problem has already been extensively investigated. For instance, Rudemo [34] studied data-driven histogram and kernel estimates based on the cross-validation method. Kernel estimates were also studied by Kutoyants [29] but in a non-adaptive framework. Donoho [14] fitted the universal thresholding procedure proposed by Donoho and Johnstone [16] for estimating Poisson intensity by using the Anscombe's transform. Kolaczyk [27] refined this idea by investigating the tails of the distribution of the noisy wavelet coefficients of the intensity. Still in the wavelet setting, Kim and Koo [25] studied maximum likelihood type estimates on sieves for an exponential family of wavelets.

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And for a particular inverse problem, Cavalier and Koo [10] first derived optimal estimates in the minimax setting. More precisely, for their tomographic problem, Cavalier and Koo [10] pointed out minimax thresholding rules on Besov balls. By using model selection, other optimal estimators have been proposed by Reynaud-Bouret [31] who obtained oracle type inequalities and minimax rates on a particular class of Besov spaces. In the more general setting of point measure, let us mention the work by Baraud and Birgé [4] which deals with histogram selection with the use of Hellinger distance. These model selection results have been generalized by Birgé [6] who applied a general methodology based on  $T$ -estimators whose performance is measured by the Hellinger distance. However, as explained by Birgé [6], this methodology is too computationally intensive to be implemented. Related works in other settings are worth citing. For instance, in Poisson regression, Kolaczyk and Nowak [28] considered penalized maximum likelihood estimates, whereas Antoniadis *et al.* [2] and Antoniadis and Sapatinas [3] focused on wavelet shrinkage.

For our purpose, it is capital to note that in the previous works, estimation is performed by assuming that the intensity has in practice a compact support known by the statistician,  $[0, 1]$  in general. Actually, procedures of previous works are used after preprocessing. The support is indeed assumed to be in  $[0, M]$ , where  $M$  is a known constant given either by some extra-knowledge concerning the data or by the largest observation. Then, all the observations are rescaled by dividing by  $M$  so that observations belong to  $[0, 1]$ . But all the previous estimators depend on a tuning parameter, which therefore depends in practice on  $M$ . If  $M$  is overestimated, the estimation is poor. Even taking the largest observation can be too rough if the distribution is heavy-tailed so that the largest observation may be very far away from the main part of the intensity. These problems become more crucial if one deals with data coming from other more complex point processes (see [19] or [32]) where one knows that the support is overestimated by the theory and where the classical trick of using the largest observation cannot be considered. Consequently the assumption of known and bounded support is not considered in the present paper.

Let us now describe more precisely our framework. We begin by giving the definition of a Poisson process to fix notations.

**Definition 1.** *Let  $N$  be a random countable subset of  $\mathbb{R}$ .  $N$  is said to be a Poisson process on  $\mathbb{R}$  if*

- *for all  $A \subset \mathbb{R}$ , the number of points of  $N$  lying in  $A$  is a random variable, denoted  $N_A$ , which obeys a Poisson law with parameter denoted by  $\mu(A)$  where  $\mu$  is a measure on  $\mathbb{R}$ ,*
- *for all finite family of disjoint sets  $A_1, \dots, A_n$ ,  $N_{A_1}, \dots, N_{A_n}$  are independent.*

The measure  $\mu$ , called the mean measure of  $N$ , is assumed to be finite to obtain almost surely a finite set of points for  $N$ . We denote by  $dN$  the discrete random measure  $\sum_{T \in N} \delta_T$  so we have for any function  $g$ ,

$$\int g(x) dN_x = \sum_{T \in N} g(T).$$

We assume that the mean measure is absolutely continuous with respect to the Lebesgue measure and for  $n$ , a fixed integer, we denote by  $f$  the intensity function of  $N$  defined by

$$\forall x \in \mathbb{R}, \quad f(x) = \frac{\mu(dx)}{n dx}.$$

We are interested in estimating  $f$  knowing the almost surely finite set of points  $N$ . The parameter  $n$  is introduced to derive results in an asymptotic setting where  $f$  is held fixed and  $n$  goes to  $+\infty$ . Furthermore, note that observing the  $n$ -sample of Poisson processes  $(N_1, \dots, N_n)$  with common

intensity  $f$  with respect to the Lebesgue measure is equivalent to observe the cumulative Poisson process  $N = \cup_{i=A}^n N_i$  with intensity  $n \times f$  with respect to the Lebesgue measure. And in addition, this setting is close to the problem of density estimation where we observe a  $n$ -sample with density  $f / \int f(x)dx$ .

Our goal is to build constructive data-driven estimators of  $f$  and for this purpose, we consider thresholding rules whose risk is measured under the  $\mathbb{L}_2$ -loss. Our framework is the following. Of course,  $f$  is non-negative and since we assume that  $\mu(\mathbb{R}) < \infty$ , this implies that  $f \in \mathbb{L}_1$ . Since we consider the  $\mathbb{L}_2$ -loss,  $f$  is assumed to be in  $\mathbb{L}_2$ . In particular,  $f$  is not assumed to be bounded (except in the minimax setting) and, as said previously, its support may be infinite.

In a different setting, the problem of estimating a density with infinite support has been partly solved from the minimax point of view. See [8] where minimax results for a class of functions depending on a gauge are established or [21] and [18] for Sobolev classes. In these papers, the loss function depends on the parameters of the functional class. Similarly, Donoho *et al.* [17] proved the optimality of wavelet linear estimators on Besov spaces  $\mathcal{B}_{p,q}^\alpha$  when the  $\mathbb{L}_p$ -risk is considered. First general results where the loss is independent of the functional class have been pointed out by Juditsky and Lambert-Lacroix [24] who investigated minimax rates on the particular class of the Besov spaces  $\mathcal{B}_{\infty,\infty}^\alpha$  for the  $\mathbb{L}_p$ -risk. When  $p > 2 + 1/\alpha$ , the minimax risk is bounded by  $(\log(n)/n)^{2\alpha/(1+2\alpha)}$  so is of the same order up to a logarithmic term as in the equivalent estimation problem on  $[0, 1]$ . However, the behavior of the minimax risk changes dramatically when  $p \leq 2 + 1/\alpha$ , and in this case, it depends on  $p$ . In addition, Juditsky and Lambert-Lacroix [24] pointed out a data-driven thresholding procedure achieving minimax rates up to a logarithmic term. In the maxiset setting, this procedure has been studied by Autin [1] and compared to other classical thresholding procedures. Finally, we can also mention that Bunea *et al.* [9] established oracle inequalities without any support assumption by using Lasso-type estimators.

## 1.2 The estimation procedure

Now, let us describe the estimation procedure considered in our paper. For this purpose, we assume in the following that the function  $f$  can be written as follows:

$$f = \sum_{\lambda \in \Lambda} \beta_\lambda \tilde{\varphi}_\lambda, \quad \text{with } \beta_\lambda = \int f(x) \varphi_\lambda(x) dx \quad (1.1)$$

where  $(\tilde{\varphi}_\lambda)_{\lambda \in \Lambda}$  and  $(\varphi_\lambda)_{\lambda \in \Lambda}$  are two infinite families of linearly independent functions of  $\mathbb{L}_2$ . Most of the further results are valid by taking  $(\tilde{\varphi}_\lambda)_{\lambda \in \Lambda} = (\varphi_\lambda)_{\lambda \in \Lambda}$  to be an orthonormal basis of  $\mathbb{L}_2$  (the Haar basis for instance). However, minimax results are established by considering special cases of biorthogonal wavelet bases and in this case  $(\tilde{\varphi}_\lambda)_{\lambda \in \Lambda}$  and  $(\varphi_\lambda)_{\lambda \in \Lambda}$  are different (see Section 3). We note

$$\|f\|_{\tilde{\varphi}} = \left( \sum_{\lambda \in \Lambda} \beta_\lambda^2 \right)^{1/2}$$

which is equal to the  $\mathbb{L}_2$ -norm of  $f$  if  $(\tilde{\varphi}_\lambda)_{\lambda \in \Lambda}$  is orthonormal. We consider thresholding estimators based on observations  $(\hat{\beta}_\lambda)_{\lambda \in \Gamma_n}$ , where  $\Gamma_n$  is a subset of  $\Lambda$  chosen later and

$$\forall \lambda \in \Lambda, \quad \hat{\beta}_\lambda = \frac{1}{n} \int_{\mathbb{R}} \varphi_\lambda(x) dN_x.$$

Observe that  $\forall \lambda \in \Lambda$ ,  $\hat{\beta}_\lambda$  is an unbiased estimator of  $\beta_\lambda$ . As Juditsky and Lambert-Lacroix [24], we threshold  $\hat{\beta}_\lambda$  according to a random positive function of  $\lambda$  depending on  $n$  and on a fixed parameter

$\gamma$  fixed later, denoted by  $\eta_{\lambda,\gamma}$  and the thresholding estimator of  $f$  is

$$\tilde{f}_{n,\gamma} = \sum_{\lambda \in \Gamma_n} \tilde{\beta}_\lambda \tilde{\varphi}_\lambda, \quad (1.2)$$

where

$$\forall \lambda \in \Lambda, \quad \tilde{\beta}_\lambda = \hat{\beta}_\lambda 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}}.$$

In the sequel, we denote  $\tilde{f}_\gamma = (\tilde{f}_{n,\gamma})_n$ .

The procedure (1.2) can also be seen as a model selection procedure. Indeed, for all  $g = \sum_{\lambda \in \Lambda} \alpha_\lambda \tilde{\varphi}_\lambda$ , we define the least square contrast by

$$\gamma_n(g) = -2 \sum_{\lambda \in \Lambda} \alpha_\lambda \hat{\beta}_\lambda + \sum_{\lambda \in \Lambda} \alpha_\lambda^2.$$

For all subset of indices  $m$ , we denote by  $S_m$  the subspace generated by  $\{\tilde{\varphi}_\lambda, \lambda \in m\}$ . The projection estimator onto  $S_m$  is defined by

$$\hat{f}_m = \arg \min_{g \in S_m} \gamma_n(g) = \sum_{\lambda \in m} \hat{\beta}_\lambda \tilde{\varphi}_\lambda.$$

Note that

$$\gamma_n(\hat{f}_m) = - \sum_{\lambda \in m} \hat{\beta}_\lambda^2.$$

If we set

$$\text{pen}(m) = \sum_{\lambda \in m} \eta_{\lambda,\gamma}^2,$$

then the thresholding estimator can be seen as a penalized projection estimator since we have

$$\tilde{f}_{n,\gamma} = \hat{f}_{\hat{m}} = \sum_{\lambda \in \Gamma_n} \hat{\beta}_\lambda 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} \tilde{\varphi}_\lambda$$

with

$$\hat{m} = \arg \min_{m \subset \Gamma_n} \left[ \gamma_n(\hat{f}_m) + \text{pen}(m) \right]. \quad (1.3)$$

Such an interpretation is used in Section 4.1 and for the proof of the main result of this paper.

### 1.3 Overview of the paper

In this paper, our goals are threefold. First of all, we wish to derive theoretical results for the  $\mathbb{L}_2$ -risk of  $\tilde{f}_\gamma$  by using three different points of view (oracle, maxiset and minimax), then we wish to discuss precisely the choice of the threshold and finally we wish to perform some simulations.

Let us now describe our results for our first aim. Theorem 1 is the main result of the paper. With a convenient choice of the threshold and under very mild assumptions on  $\Gamma_n$ , Theorem 1 proves that the thresholding estimate  $\tilde{f}_\gamma$  satisfies an oracle type inequality. We emphasize that this result is valid under very mild assumptions on  $f$ . Indeed, classical procedures use a bound for the sup-norm of  $f$  (see [10], [17] or [31]). This is not the case here where the threshold is the sum of two terms, a purely random one that is the main term and a deterministic one (see (2.2)). The definition of the threshold is extensively discussed in Section 2. By using biorthogonal wavelet

bases, we derive from Theorem 1 the oracle inequality satisfied by  $\tilde{f}_\gamma$ . More precisely, Theorem 2 in Section 4.1 shows that  $\tilde{f}_\gamma$  achieves the same performance as the oracle estimator up to a logarithmic term which is the price to pay for adaptation. From Theorem 2, we derive the maxiset results of this paper. Let us recall that the maxiset approach consists in investigating the maximal space (*maxiset*), where a given procedure achieves a given rate of convergence. For the maxiset theory, there is no a priori functional assumption. For a given procedure, the practitioner states the desired accuracy by fixing a rate and points out all the functions that can be estimated at this rate by the procedure. Obviously, the larger the maxiset, the better the procedure. We prove in Section 4.2, that under mild conditions, the maxiset of the estimate  $\tilde{f}_\gamma$  for classical rates of the form  $(\log(n)/n)^{\alpha/(1+2\alpha)}$  is, roughly speaking, the intersection of two spaces: a weak Besov space denoted  $W_\alpha$  and the classical Besov space  $\mathcal{B}_{2,\infty}^\alpha$  (see Theorem 3 and Section 4.2 for more details). Interestingly, this maxiset result provides examples of non bounded functions that can be estimated at the rate  $(\log(n)/n)^{\alpha/(1+2\alpha)}$  when  $0 < \alpha < 1/4$  (see Proposition 1). Furthermore, we derive from the maxiset result most of the minimax results briefly described now.

As said previously, Juditsky and Lambert-Lacroix [24] established minimax rates for the problem of estimating a density with an infinite support for the particular class of Besov spaces  $\mathcal{B}_{\infty,\infty}^\alpha$  and for the  $\mathbb{L}_p$ -loss. To the best of our knowledge, minimax rates are unknown for Besov spaces  $\mathcal{B}_{p,q}^\alpha$  except for very special cases described above. Our goal is to deal with this issue in the Poisson setting and for the  $\mathbb{L}_2$ -loss. We emphasize that for the minimax setting, we assume that the function to be estimated is bounded. The results that we obtain are the following. When  $p \leq 2$ , under mild assumptions, the minimax rate of convergence associated with  $\mathcal{B}_{p,q}^\alpha$  is the classical rate  $n^{-\alpha/(1+2\alpha)}$  up to a logarithmic term. So, it is of the same order as in the equivalent estimation problem on compact sets of  $\mathbb{R}$ . Furthermore, our estimate achieves this rate up to a logarithmic term. When  $p > 2$ , using our maxiset result, we prove that this last result concerning our procedure is no more true. But we prove under mild conditions that the rate of  $\tilde{f}_\gamma$  is not larger than  $(\log(n)/n)^{\alpha/(2+2\alpha-1/p)}$  up to a constant. Note that when  $p = \infty$ ,  $(\log(n)/n)^{\alpha/(2+2\alpha)}$  is the rate pointed out by Juditsky and Lambert-Lacroix [24] for minimax estimation under the  $\mathbb{L}_2$ -loss on the space  $\mathcal{B}_{\infty,\infty}^\alpha$ . Of course, when compactly supported functions are considered,  $\tilde{f}_\gamma$  is adaptive minimax on Besov spaces  $\mathcal{B}_{p,q}^\alpha$  up to a logarithmic term.

The second goal of the paper is to discuss the choice of the threshold. The starting point of this discussion is as follows. The main term of the threshold is  $(2\gamma\log(n)\tilde{V}_{\lambda,n})^{1/2}$  where  $\tilde{V}_{\lambda,n}$  is an estimate of the variance of  $\hat{\beta}_\lambda$  and  $\gamma$  is a constant to be calibrated (see (2.2) for further details). As usual,  $\gamma$  has to be large enough to obtain the theoretical results (see Theorem 1). Such an assumption is very classical (see for instance [24], [17], [10] or [1]). But, as illustrated by Juditsky and Lambert-Lacroix [24], it is often too conservative for practical issues. In this paper, the assumption on the constant  $\gamma$  is as less conservative as possible and actually most of the results are valid if  $\gamma > 1$ . So, the first issue is the following: what happens if  $\gamma \leq 1$ ? Theorem 8 of Section 5 proves that the rate obtained for estimating the simple function  $1_{[0,1]}$  is larger than  $n^{-(\gamma+\varepsilon)/2}$  for any  $\varepsilon > 0$ . This proves that  $\gamma < 1$  is a bad choice since, with  $\gamma > 1$ , we achieve the parametric rate up to a logarithmic term. Finally we consider a special class of intensity functions denoted  $\mathcal{F}_n$ . Theorems 9 and 10 provide upper and lower bounds of the maximal ratio on  $\mathcal{F}_n$  of the risk of  $\tilde{f}_\gamma$  by the oracle risk and prove that  $\gamma$  should not be too large.

Finally we validate the previous range of  $\gamma$  and refine it through a simulation study so that one can claim that  $\gamma = 1$  is a fairly good choice for all the encountered situations (finite/infinite support, bounded/unbounded intensity, smooth/non-smooth functions).

## 1.4 Outlines

The paper is organized as follows. In Section 2, the main result of this paper is established. Then, Section 3 introduces biorthogonal wavelet bases that are used to give oracle, maxiset and minimax results pointed out in Section 4. Section 5 discusses the choice of the threshold, whereas Section 6 provides some simulations. Finally, Section 7 gives the proof of the theoretical results.

## 2 The main result

In the sequel, for  $R > 0$ , if  $\mathcal{F}$  is a given Banach space, we denote  $\mathcal{F}(R)$  the ball of radius  $R$  associated with  $\mathcal{F}$ . For any  $1 \leq p \leq \infty$ , we denote

$$\|g\|_p = \left( \int |g(x)|^p dx \right)^{\frac{1}{p}}$$

with the usual modification for  $p = \infty$ . To state the main result, let us introduce the following notations that are used throughout the paper. We set

$$\forall \lambda \in \Lambda, \quad \hat{V}_{\lambda,n} = \int_{\mathbb{R}} \frac{\varphi_{\lambda}^2(x)}{n^2} dN_x,$$

the natural estimate of  $V_{\lambda,n}$  that is the variance of  $\hat{\beta}_{\lambda}$ :

$$\forall \lambda \in \Lambda, \quad V_{\lambda,n} = \mathbb{E}(\hat{V}_{\lambda,n}) = \frac{\sigma_{\lambda}^2}{n},$$

where

$$\forall \lambda \in \Lambda, \quad \sigma_{\lambda}^2 = \int_{\mathbb{R}} \varphi_{\lambda}^2(x) f(x) dx.$$

**Theorem 1.** *We assume that (1.1) is true and  $\Gamma_n$  is such that for  $\lambda \in \Gamma_n$ ,*

$$\|\varphi_{\lambda}\|_{\infty} \leq c_{\varphi,n} \sqrt{n}$$

*and that for all  $x \in \mathbb{R}$ ,*

$$\text{card}\{\lambda \in \Gamma_n : \varphi_{\lambda}(x) \neq 0\} \leq m_{\varphi,n} \log n, \quad (2.1)$$

*where  $c_{\varphi,n}$  and  $m_{\varphi,n}$  depend on  $n$  and on the family  $(\varphi_{\lambda})_{\lambda \in \Lambda}$ . Let  $\gamma > 1$ . We set*

$$\eta_{\lambda,\gamma} = \sqrt{2\gamma \log n \tilde{V}_{\lambda,n}} + \frac{\gamma \log n}{3n} \|\varphi_{\lambda}\|_{\infty}, \quad (2.2)$$

*where*

$$\tilde{V}_{\lambda,n} = \hat{V}_{\lambda,n} + \sqrt{2\gamma \log n \hat{V}_{\lambda,n} \frac{\|\varphi_{\lambda}\|_{\infty}^2}{n^2}} + 3\gamma \log n \frac{\|\varphi_{\lambda}\|_{\infty}^2}{n^2}$$

*and consider  $\tilde{f}_{n,\gamma}$  defined in (1.2). Then for all  $\varepsilon < \gamma - 1$  and for all  $p \geq 2$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$  and  $\frac{\gamma}{q} > 1 + \varepsilon$ ,*

$$\begin{aligned} \frac{\varepsilon}{2 + \varepsilon} \mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\varphi}^2) &\leq \mathbb{E} \left[ \inf_{m \subset \Gamma_n} \left\{ \left(1 + \frac{2}{\varepsilon}\right) \sum_{\lambda \notin m} \beta_{\lambda}^2 + \varepsilon \sum_{\lambda \in m} (\hat{\beta}_{\lambda} - \beta_{\lambda})^2 + \sum_{\lambda \in m} \eta_{\lambda,\gamma}^2 \right\} \right] + \\ &\quad + c_0(1 + \varepsilon)p^2 \|f\|_1 c_{\varphi,n}^2 m_{\varphi,n} \log(n) \left[ n^{-\frac{\gamma}{q(1+\varepsilon)}} + n^{-\frac{\gamma}{q}} (\max(\|f\|_1; 1))^{\frac{1}{q}} \right], \end{aligned}$$

*where  $c_0$  is an absolute constant.*

Note that this result is proved under very mild conditions on the decomposition of  $f$ . In particular we never use in the proof that we are working on the real line but only that the decomposition (1.1) exists. Observe also that if we use wavelet bases (see (3.1) in Section 3 below where we recall the standard wavelet setting) and if

$$\Gamma_n \subset \{\lambda = (j, k) \in \Lambda : 2^j \leq n^c\},$$

where  $c$  is a constant, then  $m_{\varphi, n}$  does not depend on  $n$  and in addition,

$$\sup_n \left[ c_{\varphi, n} n^{-(c-1)/2} \right] < \infty.$$

The threshold seems to be defined in a rather complicated manner. But, first observe that  $\forall \theta > 0$ ,  $\forall \lambda \in \Gamma_n$ ,

$$\sqrt{2\gamma \log(n) \hat{V}_{\lambda, n}} + \frac{\gamma \log(n)}{3n} \|\varphi_\lambda\|_\infty \leq \eta_{\lambda, \gamma} \leq c_{1, \theta} \sqrt{2\gamma \log(n) \hat{V}_{\lambda, n}} + c_{2, \theta} \frac{\gamma \log(n)}{3n} \|\varphi_\lambda\|_\infty, \quad (2.3)$$

with  $c_{1, \theta} = \sqrt{1 + \frac{1}{2\theta}}$ ,  $c_{2, \theta} = (3\sqrt{2\theta + 6} + 1)$ .

Since  $\hat{V}_{\lambda, n}$  is the natural estimate of  $V_{\lambda, n}$ , the first term of the left hand side of (2.3) is similar to the threshold introduced by Juditsky and Lambert-Lacroix [24] in the density estimation setting. But unlike Juditsky and Lambert-Lacroix [24], we add a deterministic term that allows to consider  $\gamma$  close to 1 and to control large deviations terms. In addition, since  $\eta_{\lambda, \gamma}$  cannot be equal to 0, this allows to deal with very irregular functions. However, observe that, most of the time, the deterministic term is negligible compared to the first term as soon as  $\lambda \in \Gamma_n$  satisfies  $\|\varphi_\lambda\|_\infty = o_n(n^{1/2})$ . Finally, in the same spirit,  $V_{\lambda, n}$  is slightly overestimated and we consider  $\hat{V}_{\lambda, n}$  instead of  $\tilde{V}_{\lambda, n}$  to define the threshold.

The result of Theorem 1 is an oracle type inequality. By exchanging the expectation and the infimum, the result provides the expected oracle inequality claimed in Theorem 2 of Section 4.1. Theorem 2 is derived from Theorem 1 by evaluating  $\mathbb{E}(\sum_{\lambda \in \Gamma_n} \eta_{\lambda, \gamma}^2)$  and by using biorthogonal wavelet bases.

### 3 Biorthogonal wavelet bases and Besov spaces

In this paper, the intensity  $f$  to be estimated is assumed to belong to  $\mathbb{L}_1 \cap \mathbb{L}_2$ . In this case,  $f$  can be decomposed on the Haar wavelet basis and this property is used throughout this paper. However, in Section 4.3, the Haar basis that suffers from lack of regularity is not considered. Instead, we consider a particular class of biorthogonal wavelet bases that are described now. For this purpose, let us set

$$\phi = 1_{[0, 1]}.$$

For any  $r \geq 0$ , there exist three functions  $\psi$ ,  $\tilde{\phi}$  and  $\tilde{\psi}$  with the following properties:

1.  $\tilde{\phi}$  and  $\tilde{\psi}$  are compactly supported,
2.  $\tilde{\phi}$  and  $\tilde{\psi}$  belong to  $C^{r+1}$ , where  $C^{r+1}$  denotes the Hölder space of order  $r+1$ ,
3.  $\psi$  is compactly supported and is a piecewise constant function,
4.  $\psi$  is orthogonal to polynomials of degree no larger than  $r$ ,

5.  $\{(\phi_k, \psi_{j,k})_{j \geq 0, k \in \mathbb{Z}}, (\tilde{\phi}_k, \tilde{\psi}_{j,k})_{j \geq 0, k \in \mathbb{Z}}\}$  is a biorthogonal family:  $\forall j, j' \geq 0, \forall k, k' \in \mathbb{Z}$ ,

$$\int_{\mathbb{R}} \psi_{j,k}(x) \tilde{\phi}_{k'}(x) dx = \int_{\mathbb{R}} \phi_k(x) \tilde{\psi}_{j',k'}(x) dx = 0,$$

$$\int_{\mathbb{R}} \phi_k(x) \tilde{\phi}_{k'}(x) dx = 1_{k=k'}, \quad \int_{\mathbb{R}} \psi_{j,k}(x) \tilde{\psi}_{j',k'}(x) dx = 1_{j=j', k=k'},$$

where for any  $x \in \mathbb{R}$  and for any  $(j, k) \in \mathbb{Z}^2$ ,

$$\phi_k(x) = \phi(x - k), \quad \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$$

and

$$\tilde{\phi}_k(x) = \tilde{\phi}(x - k), \quad \tilde{\psi}_{j,k}(x) = 2^{j/2} \tilde{\psi}(2^j x - k).$$

This implies that for any  $f \in \mathbb{L}_1 \cap \mathbb{L}_2$ , for any  $x \in \mathbb{R}$ ,

$$f(x) = \sum_{k \in \mathbb{Z}} \alpha_k \tilde{\phi}_k(x) + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} \beta_{j,k} \tilde{\psi}_{j,k}(x),$$

where for any  $j \geq 0$  and any  $k \in \mathbb{Z}$ ,

$$\alpha_k = \int_{\mathbb{R}} f(x) \phi_k(x) dx, \quad \beta_{j,k} = \int_{\mathbb{R}} f(x) \psi_{j,k}(x) dx.$$

Such biorthogonal wavelet bases have been built by Cohen *et al.* [11] as a special case of spline systems (see also the elegant equivalent construction of Donoho [15] from boxcar functions). Of course, recall that all these properties except the second and the forth ones are true for the Haar basis, where  $\tilde{\phi} = \phi$  and  $\tilde{\psi} = \psi = 1_{[0,1/2]} - 1_{[1/2,1]}$ , which allows to obtain in addition an orthonormal basis. This last point is not true for general biorthogonal wavelet bases but we have the frame property: there exist two constants  $c_1$  and  $c_2$  only depending on the basis such that

$$c_1 \left( \sum_{k \in \mathbb{Z}} \alpha_k^2 + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} \beta_{j,k}^2 \right) \leq \|f\|_2^2 \leq c_2 \left( \sum_{k \in \mathbb{Z}} \alpha_k^2 + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} \beta_{j,k}^2 \right).$$

In the sequel, when wavelet bases are used, we set

$$\Lambda = \{\lambda = (j, k) : j \geq -1, k \in \mathbb{Z}\}. \quad (3.1)$$

We denote for any  $\lambda \in \Lambda$ ,  $\varphi_\lambda = \phi_k$  (respectively  $\tilde{\varphi}_\lambda = \tilde{\phi}_k$ ) if  $\lambda = (-1, k)$  and  $\varphi_\lambda = \psi_{j,k}$  (respectively  $\tilde{\varphi}_\lambda = \tilde{\psi}_{j,k}$ ) if  $\lambda = (j, k)$  with  $j \geq 0$ . Similarly,  $\beta_\lambda = \alpha_k$  if  $\lambda = (-1, k)$  and  $\beta_\lambda = \beta_{j,k}$  if  $\lambda = (j, k)$  with  $j \geq 0$ . So, (1.1) is valid. An important feature of the bases introduced previously is the following: there exists a constant  $\mu_\psi > 0$  such that

$$\inf_{x \in [0,1]} |\phi(x)| \geq 1, \quad \inf_{x \in \text{supp}(\psi)} |\psi(x)| \geq \mu_\psi, \quad (3.2)$$

where  $\text{supp}(\psi) = \{x \in \mathbb{R} : \psi(x) \neq 0\}$ . This property is used throughout the paper.

Now, let us recall some properties of Besov spaces that are extensively used in the next section. We refer the reader to [13] and [20] for the definition of Besov spaces, denoted  $\mathcal{B}_{p,q}^\alpha$  in the sequel, and



a review of their properties explaining their important role in approximation theory and statistics. We just recall the sequential characterization of Besov spaces by using the biorthogonal wavelet basis (for further details, see [12]). Let  $1 \leq p, q \leq \infty$  and  $0 < \alpha < r + 1$ , the  $\mathcal{B}_{p,q}^\alpha$ -norm of  $f$  is equivalent to the norm

$$\|f\|_{\alpha,p,q} = \begin{cases} \|(\alpha_k)_k\|_{\ell_p} + \left[ \sum_{j \geq 0} 2^{jq(\alpha+1/2-1/p)} \|(\beta_{j,k})_k\|_{\ell_p}^q \right]^{1/q} & \text{if } q < \infty, \\ \|(\alpha_k)_k\|_{\ell_p} + \sup_{j \geq 0} 2^{j(\alpha+1/2-1/p)} \|(\beta_{j,k})_k\|_{\ell_p} & \text{if } q = \infty. \end{cases}$$

We use this norm to define the radius of Besov balls. For any  $R > 0$ , if  $0 < \alpha' \leq \alpha < r + 1$ ,  $1 \leq p \leq p' \leq \infty$  and  $1 \leq q \leq q' \leq \infty$ , we obviously have

$$\mathcal{B}_{p,q}^\alpha(R) \subset \mathcal{B}_{p,q'}^\alpha(R), \quad \mathcal{B}_{p,q}^\alpha(R) \subset \mathcal{B}_{p',q}^{\alpha'}(R).$$

Moreover

$$\mathcal{B}_{p,q}^\alpha(R) \subset \mathcal{B}_{p',q}^{\alpha'}(R) \text{ if } \alpha - \frac{1}{p} \geq \alpha' - \frac{1}{p'}. \quad (3.3)$$

The class of Besov spaces  $\mathcal{B}_{p,\infty}^\alpha$  provides a useful tool to classify wavelet decomposed signals in function of their regularity and sparsity properties (see [23]). Roughly speaking, regularity increases when  $\alpha$  increases whereas sparsity increases when  $p$  decreases. Especially, the spaces with indices  $p < 2$  are of particular interest since they describe very wide classes of inhomogeneous but *sparse functions* (i.e. with a few number of significant coefficients). The case  $p \geq 2$  is typical of *dense functions*.

## 4 Oracle, maxiset and minimax results

Along this section, we use biorthogonal wavelet bases as defined in Section 3.

### 4.1 Oracle inequalities

Ideal adaptation is studied in [16] using the class of shrinkage rules in the context of wavelet function estimation. This is the performance that can be achieved with the aid of an oracle. In our setting, the oracle does not tell us the true function, but tells us, for our thresholding method, the coefficients that have to be kept. This “estimator” obtained with the aid of an oracle is not a true estimator, of course, since it depends on  $f$ . But it represents an ideal for a particular estimation method. The approach of ideal adaptation is to derive true estimators which can essentially “mimic” the performance of the oracle estimator. So, using the interpretation of thresholding rules as model selection rules, the oracle provides the model  $\bar{m} \subset \Gamma_n$  such that the quadratic risk of  $\hat{f}_{\bar{m}}$  is minimum. Since, we have for any  $m \subset \Gamma_n$ ,

$$\mathbb{E}(\|\hat{f}_m - f\|_{\tilde{\varphi}}^2) = \sum_{\lambda \in m} V_{\lambda,n} + \sum_{\lambda \notin m} \beta_\lambda^2,$$

the oracle estimator  $\hat{f}_{\bar{m}}$  is obtained by taking  $\bar{m} = \{\lambda \in \Gamma_n : \beta_\lambda^2 > V_{\lambda,n}\}$  and

$$\hat{f}_{\bar{m}} = \sum_{\lambda \in \Gamma_n} \hat{\beta}_\lambda 1_{\beta_\lambda^2 > V_{\lambda,n}} \tilde{\varphi}_\lambda.$$

Its risk (the oracle risk) is then

$$\mathbb{E}(\|\hat{f}_{\bar{m}} - f\|_{\tilde{\varphi}}^2) = \mathbb{E} \sum_{\lambda \in \Gamma_n} (\hat{\beta}_\lambda 1_{\beta_\lambda^2 > V_{\lambda,n}} - \beta_\lambda)^2 + \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 = \sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n}) + \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2.$$

Our aim is now to compare the risk of  $\tilde{f}_{n,\gamma}$  to the oracle risk. We deduce from Theorem 1 the following result.

**Theorem 2.** *Let us fix two constants  $c \geq 1$  and  $c' \in \mathbb{R}$ , and let us define for any  $n$ ,  $j_0 = j_0(n)$  the integer such that  $2^{j_0} \leq n^c (\log(n))^{c'} < 2^{j_0+1}$ . Let  $\gamma > c$  and let  $\eta_{\lambda,\gamma}$  be as in Theorem 1. Then  $\tilde{f}_{n,\gamma}$  defined with*

$$\Gamma_n = \{\lambda = (j, k) \in \Lambda : j \leq j_0\}$$

*achieves the following oracle inequality:*

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \leq C_1(\gamma, \varphi) \left[ \sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n} \log(n)) + \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \right] + \frac{C_2(\gamma, \|f\|_1, c, c', \varphi)}{n} \quad (4.1)$$

where  $C_1(\gamma, \varphi)$  is a positive constant depending only on the basis and of the value of  $\gamma$  and where  $C_2(\gamma, \|f\|_1, c, c', \varphi)$  is also a positive constant depending on  $\gamma$  and the basis but also on  $\|f\|_1$ ,  $c$  and  $c'$ .

The oracle inequality (4.1) satisfied by  $\tilde{f}_{n,\gamma}$  proves that this estimator achieves essentially the oracle risk up to a logarithmic term. This logarithmic term is the price we pay for adaptivity, i.e. for not knowing the wavelet coefficients that have to be kept. In section 5, optimization of the constants of the stated result is performed for a particular class of functions.

## 4.2 Maxiset results

As said in the introduction, if  $f^*$  is a given procedure, the maxiset study of  $f^*$  consists in deciding the accuracy of the estimate by fixing a prescribed rate  $\rho^*$  and in pointing out all the functions  $f$  such that  $f$  can be estimated by the procedure  $f^*$  at the target rate  $\rho^*$ . The maxiset of the procedure  $f^*$  for this rate  $\rho^*$  is the set of all these functions. So, we set the following definition.

**Definition 2.** *Let  $\rho^* = (\rho_n^*)_n$  be a decreasing sequence of positive real numbers and let  $f^* = (f_n^*)_n$  be an estimation procedure. The maxiset of  $f^*$  associated with the rate  $\rho^*$  and the  $\mathbb{L}_2$ -loss is*

$$MS(f^*, \rho^*) = \left\{ f \in \mathbb{L}_1 \cap \mathbb{L}_2 : \sup_n [(\rho_n^*)^{-2} \mathbb{E} \|f_n^* - f\|_{\tilde{\varphi}}^2] < +\infty \right\},$$

*the ball of radius  $R > 0$  of the maxiset is defined by*

$$MS(f^*, \rho^*)(R) = \left\{ f \in \mathbb{L}_1 \cap \mathbb{L}_2 : \sup_n [(\rho_n^*)^{-2} \mathbb{E} \|f_n^* - f\|_{\tilde{\varphi}}^2] \leq R^2 \right\}.$$

To establish the maxiset result of this section, we use Theorem 2, so we need to assume that the estimation procedure is performed in a ball of  $\mathbb{L}_1 \cap \mathbb{L}_2$ . Even, if the size of the balls does not play an important role, this assumption is essential. In this setting, we use the following notation. If  $\mathcal{F}$  is a given space

$$MS(f^*, \rho^*) := \mathcal{F}$$

means in the sequel that for any  $R > 0$ , there exists  $R' > 0$  such that

$$MS(f^*, \rho^*)(R) \cap \mathbb{L}_1(R) \cap \mathbb{L}_2(R) \subset \mathcal{F}(R') \cap \mathbb{L}_1(R) \cap \mathbb{L}_2(R)$$

and for any  $R' > 0$ , there exists  $R > 0$  such that

$$\mathcal{F}(R') \cap \mathbb{L}_1(R') \cap \mathbb{L}_2(R') \subset MS(f^*, \rho^*)(R) \cap \mathbb{L}_1(R') \cap \mathbb{L}_2(R').$$

In this section, for any  $\alpha > 0$ , we investigate the set of functions that can be estimated by  $\tilde{f}_\gamma = (\tilde{f}_{n,\gamma})_n$  at the rate  $\rho_\alpha = (\rho_{n,\alpha})_n$ , where for any  $n$ ,

$$\rho_{n,\alpha} = \left( \frac{\log(n)}{n} \right)^{\frac{\alpha}{1+2\alpha}}.$$

More precisely, we investigate for any radius  $R > 0$ :

$$MS(\tilde{f}_\gamma, \rho_\alpha)(R) = \left\{ f \in \mathbb{L}_1 \cap \mathbb{L}_2 : \sup_n \left[ \rho_{n,\alpha}^{-2} \mathbb{E} \|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2 \right] \leq R^2 \right\}.$$

To characterize maxisets of  $\tilde{f}_\gamma$ , we introduce the following spaces.

**Definition 3.** We define for all  $R > 0$  and for all  $s > 0$ ,

$$W_s = \left\{ f = \sum_{\lambda \in \Lambda} \beta_\lambda \tilde{\varphi}_\lambda : \sup_{t>0} t^{\frac{-4s}{1+2s}} \sum_{\lambda \in \Lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda t} < \infty \right\},$$

the ball of radius  $R$  associated with  $W_s$  is:

$$W_s(R) = \left\{ f = \sum_{\lambda \in \Lambda} \beta_\lambda \tilde{\varphi}_\lambda : \sup_{t>0} t^{\frac{-4s}{1+2s}} \sum_{\lambda \in \Lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda t} \leq R^{\frac{2}{1+2s}} \right\},$$

and for any sequence of spaces  $\Gamma = (\Gamma_n)_n$  included in  $\Lambda$ ,

$$B_{2,\Gamma}^s = \left\{ f = \sum_{\lambda \in \Lambda} \beta_\lambda \tilde{\varphi}_\lambda : \sup_n \left[ \left( \frac{\log(n)}{n} \right)^{-2s} \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \right] < \infty \right\}$$

and

$$B_{2,\Gamma}^s(R) = \left\{ f = \sum_{\lambda \in \Lambda} \beta_\lambda \tilde{\varphi}_\lambda : \sup_n \left[ \left( \frac{\log(n)}{n} \right)^{-2s} \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \right] \leq R^2 \right\}.$$

In [13], a justification of the form of the radius of  $W_s$  and further details are provided. These spaces can be viewed as weak versions of classical Besov spaces, hence they are denoted in the sequel weak Besov spaces. In particular, the spaces  $W_s$  naturally model sparse signals (see [33]). Note that if for all  $n$ ,

$$\Gamma_n = \{\lambda = (j, k) \in \Lambda : j \leq j_0\}$$

with

$$2^{j_0} \leq \left( \frac{n}{\log n} \right)^c < 2^{j_0+1}, \quad c > 0$$

then,  $B_{2,\Gamma}^s$  is the classical Besov space  $\mathcal{B}_{2,\infty}^{s/c}$  if some properties of regularity and vanishing moments are satisfied by the wavelet basis (see Section 3). We define  $B_{2,\Gamma}^s$  and  $W_s$  by using biorthogonal wavelet bases. However, as established in [13], they also have different definitions proving that, under mild conditions, this dependence on the basis is not crucial at all. Using Theorem 2, we have the following result.

**Theorem 3.** *Let us fix two constants  $c \geq 1$  and  $c' \in \mathbb{R}$ , and let us define for any  $n$ ,  $j_0 = j_0(n)$  the integer such that  $2^{j_0} \leq n^c (\log(n))^{c'} < 2^{j_0+1}$ . Let  $\gamma > c$  and let  $\eta_{\lambda, \gamma}$  be as in Theorem 1. Then, the procedure defined in (1.2) with the sequence  $\Gamma = (\Gamma_n)_n$  such that*

$$\Gamma_n = \{\lambda = (j, k) \in \Lambda : j \leq j_0\}$$

*achieves the following maxiset performance: for all  $\alpha > 0$ ,*

$$MS(\tilde{f}_\gamma, \rho_\alpha) := B_{2, \Gamma}^{\frac{\alpha}{1+2\alpha}} \cap W_\alpha.$$

*In particular, if  $c' = -c$  and  $0 < \frac{\alpha}{c(1+2\alpha)} < r + 1$ , where  $r$  is the parameter of the biorthogonal basis introduced in Section 3,*

$$MS(\tilde{f}_\gamma, \rho_\alpha) := \mathcal{B}_{2, \infty}^{\frac{\alpha}{c(1+2\alpha)}} \cap W_\alpha.$$

**Remark 1.** *In order to obtain maxisets as large as possible, Inequality (7.7) of the proof of Theorem 3 suggests to choose  $\gamma > 1$  as small as possible.*

The maxiset of  $\tilde{f}_\gamma$  is characterized by two spaces: a weak Besov space that is directly connected to the thresholding nature of  $\tilde{f}_\gamma$  and the space  $B_{2, \Gamma}^{\alpha/(1+2\alpha)}$  that handles the coefficients that are not estimated, which corresponds to the indices  $j > j_0$ . This maxiset result is similar to the result obtained by Autin [1] in the density estimation setting but our assumptions are less restrictive (see Theorem 5.1 of [1]).

Now, let us point out a family of examples of functions that illustrates the previous result. For this purpose, we consider the Haar basis that allows to have simple formula for the wavelet coefficients. Let us consider for any  $0 < \beta < 1/2$ ,  $f_\beta$  such that

$$\forall x \in \mathbb{R}, \quad f_\beta(x) = x^{-\beta} 1_{x \in ]0, 1]}.$$

The following result points out that if  $\alpha$  is small enough, for a convenient choice of  $\beta$ ,  $f_\beta$  belongs to  $MS(\tilde{f}_\gamma, \rho_\alpha)$  (so  $f_\beta$  can be estimated at the rate  $\rho_\alpha$ ), and in addition  $f_\beta \notin \mathbb{L}_\infty$ .

**Proposition 1.** *We consider the Haar basis and we set  $c' = -c$ . For  $0 < \alpha < 1/4$ , under the assumptions of Theorem 3, if*

$$0 < \beta \leq \frac{1 - 4\alpha}{2 + 4\alpha},$$

*then for  $c$  large enough,*

$$f_\beta \in MS(\tilde{f}_\gamma, \rho_\alpha) := \mathcal{B}_{2, \infty}^{\frac{\alpha}{c(1+2\alpha)}} \cap W_\alpha,$$

*where  $\mathcal{B}_{2, \infty}^{\frac{\alpha}{c(1+2\alpha)}}$  and  $W_\alpha$  are viewed as sequence spaces. In addition,  $f_\beta \notin \mathbb{L}_\infty$ .*

This result is proved by using the Haar basis, so the functional spaces are viewed as sequence spaces. We conjecture that for more general biorthogonal wavelet bases, we can also build not bounded functions that belong to  $MS(\tilde{f}_\gamma, \rho_\alpha)$ .

### 4.3 Minimax results

Let  $\mathcal{F}$  be a functional space and  $\mathcal{F}(R)$  be the ball of radius  $R$  associated with  $\mathcal{F}$ .  $\mathcal{F}(R)$  is assumed to belong to a ball of  $\mathbb{L}_1 \cap \mathbb{L}_2$ . Let us recall that a procedure  $f^* = (f_n^*)_n$  achieves the rate  $\rho^* = (\rho_n^*)_n$  on  $\mathcal{F}(R)$  (for the  $\mathbb{L}_2$ -loss) if

$$\sup_n \left[ (\rho_n^*)^{-2} \sup_{f \in \mathcal{F}(R)} \mathbb{E}(\|f_n^* - f\|_\varphi^2) \right] < \infty.$$

Let us consider the procedure  $\tilde{f}_\gamma$  and the rate  $\rho_\alpha = (\rho_{n,\alpha})_n$  where for any  $n$ ,

$$\rho_{n,\alpha} = \left( \frac{\log(n)}{n} \right)^{\frac{\alpha}{1+2\alpha}}$$

as in the previous section. Obviously,  $\tilde{f}_\gamma$  achieves the rate  $\rho_\alpha$  on  $\mathcal{F}(R)$  if and only if there exists  $R' > 0$  such that

$$\mathcal{F}(R) \subset MS(\tilde{f}_\gamma, \rho_\alpha)(R') \cap \mathbb{L}_1(R') \cap \mathbb{L}_2(R').$$

Using results of the previous section, if  $c' = -c$  and if properties of regularity and vanishing moments are satisfied by the wavelet basis, this is satisfied if and only if there exists  $R'' > 0$  such that

$$\mathcal{F}(R) \subset \mathcal{B}_{2,\infty}^{\frac{\alpha}{c(1+2\alpha)}}(R'') \cap W_\alpha(R'') \cap \mathbb{L}_1(R'') \cap \mathbb{L}_2(R'').$$

We apply this simple rule for Besov balls. So, in the sequel, we assume that the function  $f$  to be estimated belongs to a ball of  $\mathbb{L}_1 \cap \mathbb{L}_2$ . In addition, we assume that  $f$  also belongs to a ball of  $\mathbb{L}_\infty$ . This last assumption which is not necessary to derive maxiset results (see Theorem 3 or Proposition 1) is unavoidable in some sense in the minimax setting. For a precise justification of this point, see for instance Corollary 1 of [6]. Consequently, in the sequel, we set for any  $R > 0$ ,

$$\mathcal{L}_{1,2,\infty}(R) = \{f : \|f\|_1 \leq R, \|f\|_2 \leq R, \|f\|_\infty \leq R\}.$$

In the sequel, minimax results depend on the parameter  $r$  of the biorthogonal basis introduced in Section 3 to measure the regularity of the reconstruction wavelets  $(\tilde{\phi}, \tilde{\psi})$ .

#### 4.3.1 Minimax estimation on Besov spaces $\mathcal{B}_{p,q}^\alpha$ when $p \leq 2$

To the best of our knowledge, the minimax rate is unknown for  $\mathcal{B}_{p,q}^\alpha$  when  $p < \infty$ . Let us investigate this problem by pointing out the minimax properties of  $\tilde{f}_\gamma$  on  $\mathcal{B}_{p,q}^\alpha$  when  $p \leq 2$ . We have the following result.

**Theorem 4.** *Let  $R, R' > 0$ ,  $1 \leq p, q \leq \infty$  and  $\alpha \in \mathbb{R}$  such that  $\max(0, 1/p - 1/2) < \alpha < r + 1$ . Let  $c \geq 1$  large enough such that*

$$\alpha \left( 1 - \frac{1}{c(1+2\alpha)} \right) \geq \frac{1}{p} - \frac{1}{2}. \quad (4.2)$$

*Let us define for any  $n$ ,  $j_0 = j_0(n)$  the integer such that*

$$2^{j_0} \leq n^c (\log(n))^{-c} < 2^{j_0+1}.$$

*Then, if  $p \leq 2$ ,  $\tilde{f}_\gamma = (\tilde{f}_{n,\gamma})_n$  defined with*

$$\Gamma_n = \{\lambda = (j, k) \in \Lambda : j \leq j_0\}$$

*and  $\gamma > c$  achieves the rate  $\rho_\alpha$  on  $\mathcal{B}_{p,q}^\alpha(R) \cap \mathcal{L}_{1,2,\infty}(R')$ . Indeed, for any  $n$ ,*

$$\sup_{f \in \mathcal{B}_{p,q}^\alpha(R) \cap \mathcal{L}_{1,2,\infty}(R')} \mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_\varphi^2) \leq C(\gamma, c, R, R', \alpha, p, \varphi) \left( \frac{\log n}{n} \right)^{2\alpha/(1+2\alpha)} \quad (4.3)$$

where  $C(\gamma, c, R, R', \alpha, p, \varphi)$  depends on  $R'$ ,  $\gamma$ ,  $c$ , on the parameters of the Besov ball and on the basis.

Furthermore, let  $p^* \geq 1$  and  $\alpha^* > 0$  such that

$$\alpha^* \left( 1 - \frac{1}{c(1 + 2\alpha^*)} \right) \geq \frac{1}{p^*} - \frac{1}{2}. \quad (4.4)$$

Then,  $\tilde{f}_\gamma$  is adaptive minimax up to a logarithmic term on

$$\{\mathcal{B}_{p,q}^\alpha \cap \mathcal{L}_{1,2,\infty} : \alpha^* \leq \alpha < r + 1, p^* \leq p \leq 2, 1 \leq q \leq \infty\}.$$

This result points out the minimax rate associated with  $\mathcal{B}_{p,q}^\alpha(R) \cap \mathcal{L}_{1,2,\infty}(R')$  up to a logarithmic term and in addition proves that it is of the same order as in the equivalent estimation problem on  $[0, 1]$  (see [17]). It means that, roughly speaking, it is not harder to estimate sparse non-compactly supported functions than sparse compactly supported functions from the minimax point of view. In addition, the procedure  $\tilde{f}_\gamma$  does the job up to a logarithmic term. When  $p > 2$  (i.e., when dense functions are considered), this conclusion does not remain true.

#### 4.3.2 Minimax estimation on Besov spaces $\mathcal{B}_{p,q}^\alpha$ when $p > 2$

Before considering the case of estimation of non-compactly supported functions, let us establish the following result. We denote  $\mathcal{K}$  the set of compact sets of  $\mathbb{R}$  containing a non-empty interval. We define for  $K \in \mathcal{K}$ ,  $\mathcal{B}_{p,q,K}^\alpha(R)$  the set of functions supported by  $K$  and belonging to  $\mathcal{B}_{p,q}^\alpha(R)$ .

**Corollary 1.** *We assume that assumptions of Theorem 4 are true. For any  $p \geq 1$ ,  $\tilde{f}_\gamma$  achieves the rate  $\rho_\alpha$  on  $\mathcal{B}_{p,q,K}^\alpha(R) \cap \mathcal{L}_{1,2,\infty}(R')$ .*

Furthermore,  $\tilde{f}_\gamma$  is adaptive minimax up to a logarithmic term on

$$\{\mathcal{B}_{p,q,K}^\alpha \cap \mathcal{L}_{1,2,\infty} : \alpha^* \leq \alpha < r + 1, p^* \leq p \leq \infty, 1 \leq q \leq \infty, K \in \mathcal{K}\},$$

where  $\alpha^*$  and  $p^*$  satisfy (4.4).

To prove this corollary, it is enough to apply Theorem 4 and to note that  $\mathcal{B}_{p,q,K}^\alpha(R) \subset \mathcal{B}_{p,\infty,K}^\alpha(R) \subset \mathcal{B}_{2,\infty,K}^\alpha(\tilde{R})$  for  $\tilde{R}$  large enough when  $p > 2$ .

When non-compactly supported functions are considered, this result is not true and we can prove the following theorem.

**Theorem 5.** *Let  $p > 2$  and  $\alpha > 0$ . There exists a positive function  $f$  such that*

$$f \in \mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_\infty \cap \mathcal{B}_{p,\infty}^\alpha \text{ and } f \notin W_\alpha,$$

where the function spaces are viewed as sequential spaces (the Haar basis is used).

**Remark 2.** *This result is established by using the Haar basis. We conjecture that it remains true for more general biorthogonal wavelet bases.*

This result proves that  $\tilde{f}_\gamma$  does not achieve the rate  $\rho_\alpha$  on  $\mathcal{B}_{p,\infty}^\alpha$  when  $p > 2$ , showing that minimax statements of Section 4.3.1 are not valid in this setting. As said previously, it seems to us that minimax rates and adaptive minimax rates are unknown for  $\mathcal{B}_{p,\infty}^\alpha$ , when  $2 < p < \infty$  even if Donoho *et al.* [17] provided some lower bounds in the density framework. For the case  $p = \infty$ , see [24].

Now, let us investigate the rate achieved by  $\tilde{f}_\gamma$  on  $\mathcal{B}_{p,q}^\alpha(R)$  when  $p > 2$ .

**Theorem 6.** *Let  $R, R' > 0$ ,  $1 \leq q \leq \infty$ ,  $2 < p \leq \infty$  and  $\alpha \in \mathbb{R}$  such that  $1/(2p) < \alpha < r + 1$ . Let us define for any  $n$ ,  $j_0 = j_0(n)$  the integer such that*

$$2^{j_0} \leq n^c (\log n)^{-c} < 2^{j_0+1},$$

*with  $c \geq 1$ . Then,  $\tilde{f}_\gamma = (\tilde{f}_{n,\gamma})_n$  defined with*

$$\Gamma_n = \{\lambda = (j, k) \in \Lambda : j \leq j_0\}$$

*and  $\gamma > c$  achieves the following performance. For any  $n$ ,*

$$\sup_{f \in \mathcal{B}_{p,q}^\alpha(R) \cap \mathcal{L}_{1,2,\infty}(R')} \mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \leq C(\gamma, c, R, R', \alpha, p, \varphi) \left( \frac{\log n}{n} \right)^{\frac{\alpha}{1+\alpha-\frac{1}{2p}}}.$$

*where  $C(\gamma, c, R, R', \alpha, p, \varphi)$  depends on  $R'$ ,  $\gamma$ ,  $c$ ,  $c'$ , on the parameters of the Besov ball and on the basis.*

Note that when  $p = \infty$ , the risk is bounded by  $\left(\frac{\log n}{n}\right)^{\frac{\alpha}{1+\alpha}}$  up to a constant, which is the rate of the minimax risk on  $\mathcal{B}_{\infty,\infty}^\alpha(R)$  up to a logarithmic term in the density estimation setting (see Theorem 1 of [24]). However,  $\frac{\alpha}{1+\alpha-\frac{1}{2p}} \xrightarrow{p \rightarrow 2} \frac{\alpha}{\alpha+\frac{3}{4}}$  and  $\left(\frac{\log n}{n}\right)^{\frac{\alpha}{\alpha+\frac{3}{4}}} \gg \rho_{n,\alpha}^2$ . So,  $\tilde{f}_\gamma$  is probably not adaptive minimax on the whole class of Besov spaces. However, we establish that our procedure is adaptive minimax (with the exact power of the logarithmic factor) over weak Besov spaces without any support assumption.

#### 4.3.3 Minimax estimation on $W_\alpha$ and adaptation with respect to $\alpha$

We investigate in this section a lower bound for the minimax risk on  $W_\alpha(R) \cap \mathcal{B}_{2,\infty}^{\frac{\alpha}{1+2\alpha}}(R') \cap \mathcal{L}_{1,2,\infty}(R'')$  for  $R, R', R'' > 0$  viewed as sequence spaces for the Haar basis and we set

$$\mathcal{R}(W_\alpha(R) \cap \mathcal{B}_{2,\infty}^{\frac{\alpha}{1+2\alpha}}(R') \cap \mathcal{L}_{1,2,\infty}(R'')) = \inf_{\hat{f}} \sup_{f \in W_\alpha(R) \cap \mathcal{B}_{2,\infty}^{\frac{\alpha}{1+2\alpha}}(R') \cap \mathcal{L}_{1,2,\infty}(R'')} \mathbb{E}(\|\hat{f} - f\|_{\tilde{\varphi}}^2).$$

**Theorem 7.** *For  $\alpha > 0$ , we have*

$$\liminf_{n \rightarrow \infty} \rho_{n,\alpha}^{-2} \mathcal{R}(W_\alpha(R) \cap \mathcal{B}_{2,\infty}^{\frac{\alpha}{1+2\alpha}}(R') \cap \mathcal{L}_{1,2,\infty}(R'')) \geq c(\alpha) R^{\frac{2}{1+2\alpha}},$$

*where  $c(\alpha)$  depends only on  $\alpha$ , as soon as  $R'' \geq 1$  and  $R' \geq R^{\frac{1}{1+2\alpha}} \geq 1$ .*

Using Theorem 3, we immediately deduce the following result.

**Corollary 2.** *The procedure  $\tilde{f}_\gamma$  defined in Theorem 4 with  $c = -c' = 1$  and with  $\gamma > 1$  is minimax on  $W_\alpha(R) \cap \mathcal{B}_{2,\infty}^{\frac{\alpha}{1+2\alpha}}(R') \cap \mathcal{L}_{1,2,\infty}(R'')$  and is adaptive minimax on*

$$\left\{ W_\alpha(R) \cap \mathcal{B}_{2,\infty}^{\frac{\alpha}{1+2\alpha}}(R') \cap \mathcal{L}_{1,2,\infty}(R'') : \alpha > 0, 1 \leq R'', 1 \leq R \leq R' \right\}.$$

**Remark 3.** *These results are established for the Haar basis. It is probably true for more general biorthogonal wavelet bases, but we were not able to prove it.*

## 5 How to choose the parameter $\gamma$

In this section, our goal is to find lower and upper bounds for the parameter  $\gamma$ . The aim and proofs are inspired by Birgé and Massart [7] who considered penalized estimators and calibrated constants for penalties in a Gaussian regression framework. In particular, they showed that if the penalty constant is smaller than 1, then the penalized estimator behaves in a quite unsatisfactory way. This study was used in practice to derive adequate data-driven penalties by Lebarbier [30].

We assume that the function  $f$  to be estimated belongs to a restricted functional space. More precisely, we assume that for  $n$  large enough,  $f$  belongs to  $\mathcal{F}_n$  where for any  $n$ ,

$$\mathcal{F}_n = \left\{ f \in \mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_\infty : F_\lambda \geq \frac{(\log n)(\log \log n)}{n} 1_{F_\lambda > 0}, \forall \lambda \in \Lambda \right\},$$

with  $F_\lambda = \int_{\text{supp}(\varphi_\lambda)} f(x) dx$ . Observe that  $\mathcal{F}_n$  only contains functions with finite support. If the Haar basis is considered, any function supported by  $[0, 1]$  that is constant on each interval of a dyadic partition of  $[0, 1]$  belongs to  $\mathcal{F}_n$  for  $n$  large enough. In addition, the interest of the class  $\mathcal{F}_n$  lies in the natural bridge it constitutes between the model of this paper and the regression model for which the number of non-zero coefficients is always bounded by  $n$ . These reasons justify the importance of well estimating functions of  $\mathcal{F}_n$  with an appropriate choice for  $\gamma$ . We naturally consider along this section the Haar basis and we define for any  $n$ ,  $j_0 = j_0(n)$  the integer such that  $2^{j_0} \leq n < 2^{j_0+1}$ . Then  $\tilde{f}_{n,\gamma}$  is defined with

$$\Gamma_n = \{\lambda = (j, k) \in \Lambda : j \leq j_0\}.$$

In the sequel, we prove that, roughly speaking,  $\tilde{f}_{n,\gamma}$  cannot achieve good performance from the oracle point of view if the parameter  $\gamma$  is smaller than 1 or larger than 16.

### 5.1 Lower bound for $\gamma$

In this section, we provide a lower bound for the parameter  $\gamma$ . We have the following result.

**Theorem 8.** *We estimate  $f = 1_{[0,1]} \in \mathcal{F}_n$  with  $\tilde{f}_{n,\gamma}$  such that in view of (2.3), we set*

$$\forall \lambda \in \Gamma_n, \quad \eta_{\lambda,\gamma} = \sqrt{2\gamma \log(n) \hat{V}_{\lambda,n}} + \|\varphi_\lambda\|_\infty \frac{\log(n) u_n}{n},$$

with  $(u_n)_n$  a deterministic bounded sequence. Then for all  $\varepsilon > 0$ , we obtain for any  $n$ ,

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \geq \frac{1}{n^{\gamma+\varepsilon}}(1 + o_n(1)).$$

This result shows that we need  $\gamma \geq 1$  to obtain a good convergence rate. Indeed, for any  $n$ , Theorem 2 (established with  $\gamma > 1$ ) gives the bound

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \leq C \frac{\log n}{n},$$

where  $C$  is a constant.



## 5.2 Upper bound for $\gamma$

In this section, we provide an upper bound for the parameter  $\gamma$ . In Remark 1, we have already noticed that the performances of  $\tilde{f}_\gamma$  are worse when  $\gamma$  increases. More justifications of this point are provided in this section.

**Theorem 9.** *Let  $\gamma = 1 + \sqrt{2}$  and let  $\eta_{\lambda,\gamma}$  be as in Theorem 1. Then  $\tilde{f}_{n,\gamma}$  achieves the following oracle inequality: for  $n$  large enough,*

$$\sup_{f \in \mathcal{F}_n} \frac{\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2)}{\sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n}) + \frac{1}{n}} \leq 12 \log n.$$

Now, let us assume that for a choice of  $\gamma$ , say  $\gamma_{\min}$ , the corresponding threshold  $\eta_{\lambda,\gamma_{\min}}$  leads to satisfying results (for instance, Theorem 9 tells us that  $\gamma = 1 + \sqrt{2}$  is a good choice). Then let us fix  $\gamma$  larger than  $\gamma_{\min}$  and let us consider the estimator  $\tilde{f}_{n,\gamma}$  associated with the threshold  $\eta_{\lambda,\gamma}$  as built in Theorem 1. Our goal is to obtain a lower bound of the maximal risk of  $\tilde{f}_{n,\gamma}$  on  $\mathcal{F}_n$  larger than the upper bound obtained for  $\eta_{\lambda,\gamma_{\min}}$ . This means that choosing  $\gamma$  is a bad choice. This goal is reached in the following theorem.

**Theorem 10.** *Let  $\gamma_{\min} > 1$  be fixed and let  $\gamma > \gamma_{\min}$ . We still consider the thresholding rule associated with  $\gamma$  (see Theorem 1). Then,*

$$\sup_{f \in \mathcal{F}_n} \frac{\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2)}{\sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n}) + \frac{1}{n}} \geq (\sqrt{\gamma} - \sqrt{\gamma_{\min}})^2 2 \log n (1 + o_n(1)).$$

If we choose  $\gamma_{\min} = 1 + \sqrt{2}$  and apply Theorem 9, the maximal oracle ratio of the estimator  $\tilde{f}_{n,\gamma}$  is not larger than  $12 \log n$ . So, if  $\gamma > 16$ , which yields  $(\sqrt{\gamma} - \sqrt{\gamma_{\min}})^2 > 6$ , the resulting maximal oracle ratio of  $\tilde{f}_{n,\gamma}$  is larger than  $12 \log n$ . In addition, note that the function used in Theorem 8 is also in  $\mathcal{F}_n$ . So, finally the convenient value of  $\gamma$  belongs to  $[1, 16]$ .

## 6 Simulations

In this section, some simulations are provided and the performances of the thresholding rule are measured from the numerical point of view. We also discuss the ideal choice for the parameter  $\gamma$  keeping in mind that the value  $\gamma = 1$  constitutes a border for the theoretical results (see Section 5). For these purposes, the procedure is performed for estimating various intensity signals and the wavelet set-up associated with biorthogonal wavelet bases is considered. More precisely, we focus either on the Haar basis where

$$\phi = \tilde{\phi} = 1_{[0,1]}, \quad \psi = \tilde{\psi} = 1_{[0,1/2]} - 1_{[1/2,1]}$$

or on a special case of spline systems given in Figure 1. This latter basis, called hereafter the spline basis, has the following properties. First, the support of  $\phi$ ,  $\psi$ ,  $\tilde{\phi}$  and  $\tilde{\psi}$  is included in  $[-4, 5]$ . The reconstruction wavelets  $\tilde{\phi}$  and  $\tilde{\psi}$  belong to  $C^{1.272}$ . Finally, the wavelet  $\psi$  is a piecewise constant function orthogonal to polynomials of degree 4 (see [15]). So, such a basis has properties 1–5 required in Section 3 with  $m = 0.272$ . Then, the signal  $f$  to be estimated is decomposed as follows:

$$f = \sum_{\lambda \in \Lambda} \beta_\lambda \tilde{\varphi}_\lambda = \sum_{k \in \mathbb{Z}} \beta_{-1,k} \tilde{\phi}_k + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} \beta_{j,k} \tilde{\psi}_{j,k}.$$

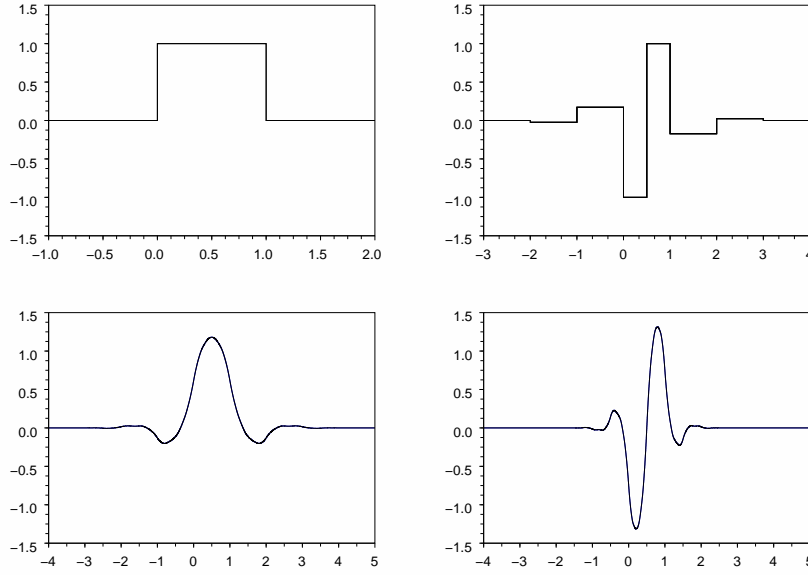


Figure 1: The spline basis. Top:  $\phi$  and  $\psi$ , Bottom:  $\tilde{\phi}$  and  $\tilde{\psi}$

For estimating  $f$ , we use the observations  $(\hat{\beta}_\lambda)_{\lambda \in \Lambda}$  associated with a Poisson process  $N$  whose intensity with respect to the Lebesgue measure is  $n \times f$ . Since  $\phi$  and  $\psi$  are piecewise constant functions, accurate values of the observations are available, which allows to avoid many computational and approximation issues that often arise in the wavelet setting. To shed light on typical aspects of Poisson intensity estimation, Figure 2 displays the reconstruction obtained by using only the coarsest noisy wavelet coefficients of a particular signal (the density of a Gaussian variable with mean 0.5 and standard deviation 0.25) with  $n = 4096$ . We mean that  $(\beta_{j,k})_{j \geq -1, k \in \mathbb{Z}}$  is estimated by  $(\hat{\beta}_{j,k})_{-1 \leq j \leq 10, k \in \mathbb{Z}}$  without using thresholding. As expected, variability highly depends on the local values of the signal. So, our framework is very different from classical regression where we observe random variables with common variance. The thresholding rule considered in this section is  $\tilde{f}_\gamma = (\tilde{f}_{n,\gamma})_n$  with  $\tilde{f}_{n,\gamma}$  defined in (1.2) with

$$\Gamma_n = \{\lambda = (j, k) : -1 \leq j \leq j_0, k \in \mathbb{Z}\}$$

and

$$\eta_{\lambda,\gamma} = \sqrt{2\gamma \log(n) \hat{V}_{\lambda,n}} + \frac{\gamma \log n}{3n} \|\varphi_\lambda\|_\infty.$$

Observe that  $\eta_{\lambda,\gamma}$  slightly differs from the threshold defined in (2.2) since  $\tilde{V}_{\lambda,n}$  is now replaced with  $\hat{V}_{\lambda,n}$ . Such a modification is natural in view of (2.3) and Theorem 8. In particular, it allows to derive the parameter  $\gamma$  as an explicit function of the threshold. We guess that the performances of our thresholding rule associated with the threshold  $\eta_{\lambda,\gamma}$  defined in (2.2) are very close. Now, to complete the definition of the estimate, we have to choose the parameters  $j_0$  and  $\gamma$ . This choice is capital and is extensively discussed in the sequel. Using  $n = 1024$ , Figure 3 displays 9 examples of intensity reconstructions obtained with  $j_0 = \log_2(n) = 10$  and  $\gamma = 1$ . These functions are respectively denoted 'Haar1', 'Haar2', 'Blocks', 'Comb', 'Gauss1', 'Gauss2', 'Beta0.5', 'Beta4' and 'Bumps' and have been chosen to represent the wide variety of signals arising in signal processing

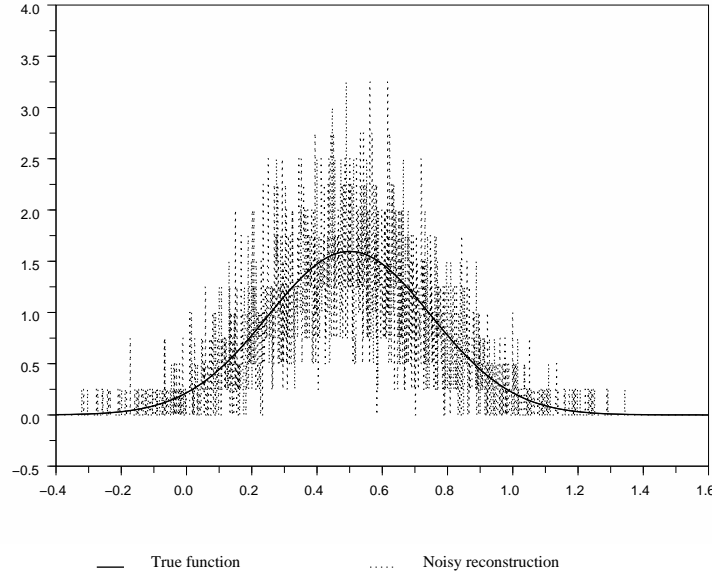


Figure 2: Plots of the signal  $f(x) = \frac{1}{0.25\sqrt{2\pi}} \exp\left(\frac{(x-0.5)^2}{2 \times 0.25^2}\right)$  and purely noisy reconstruction with  $n = 4096$  based on the wavelet coefficients until the level 10 and by using the Haar basis.

(see the Appendix for a precise definition of each signal). Each of them satisfies  $\|f\|_1 = 1$  and can be classified according to the following criteria: the smoothness, the size of the support (finite/infinite), the value of the sup norm (finite/infinite) and the shape (to be piecewise constant or a mixture of peaks). In particular, the signal 'Comb' (respectively 'Beta0.5') is inspired by the construction of the counter-example proposed in Theorem 5 (respectively Proposition 1).

More interestingly, numerical results are provided to answer the question about the choice of  $\gamma$ . Given  $n$  and a function  $f$ , we denote  $R_n(\gamma)$  the ratio between the  $\ell_2$ -performance of our procedure (depending on  $\gamma$ ) and the oracle risk where the wavelet coefficients at levels  $j > j_0$  are omitted. We have:

$$R_n(\gamma) = \frac{\sum_{\lambda \in \Gamma_n} (\tilde{\beta}_\lambda - \beta_\lambda)^2}{\sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n})} = \frac{\sum_{\lambda \in \Gamma_n} (\hat{\beta}_\lambda 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} - \beta_\lambda)^2}{\sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n})}.$$

Of course,  $R_n$  is a stepwise function and the change points of  $R_n$  correspond to the values of  $\gamma$  such that there exists  $\lambda$  with  $\eta_{\lambda,\gamma} = |\hat{\beta}_\lambda|$ . The average over 1000 simulations of  $R_n(\gamma)$  is computed providing an estimation of  $\mathbb{E}(R_n(\gamma))$ . This average ratio, denoted  $\overline{R_n}(\gamma)$  and viewed as a function of  $\gamma$ , is plotted for three signals 'Haar1', 'Gauss1' and 'Bumps' for  $n \in \{64, 128, 256, 512, 1024, 2048, 4096\}$ . For non compactly supported signals, to compute the ratio, the wavelet coefficients associated with the tails of the signals are omitted but we ensure that this approximation is negligible with respect to the values of  $R_n$ . The parameter  $j_0$  takes the value  $j_0 = \log_2(n)$ . Fixing  $j_0 = \log_2(n)$  is natural in view of Theorem 2 (applied with  $c = 1$  and  $c' = 0$ ) and Theorem 8. Figure 4 displays  $\overline{R_n}$  for 'Haar1' decomposed on the Haar basis. The left side of Figure 4 gives a general idea of the shape of  $\overline{R_n}$ , while the right side focuses on small values of  $\gamma$ . Similarly, Figures 5 and 6 display  $\overline{R_n}$  for 'Gauss1' decomposed on the spline basis and for 'Bumps' decomposed on the Haar and the spline bases.

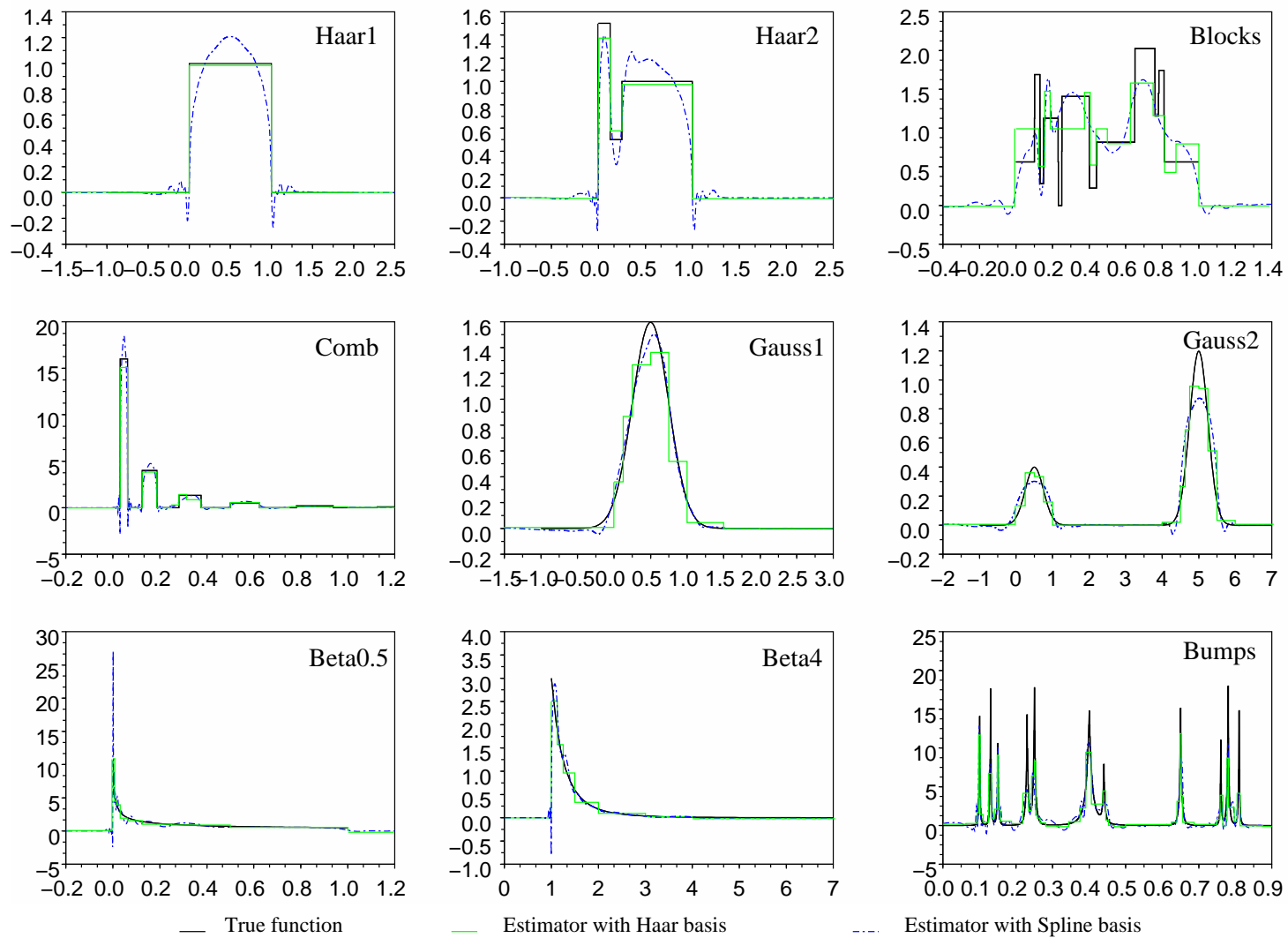


Figure 3: Reconstructions by using the Haar and the spline bases of 9 signals with  $n = 1024$ ,  $j_0 = 10$  and  $\gamma = 1$ . Top: 'Haar1', 'Haar2', 'Blocks'; Middle: 'Comb', 'Gauss1', 'Gauss2'; Bottom: 'Beta0.5', 'Beta4', 'Bumps'

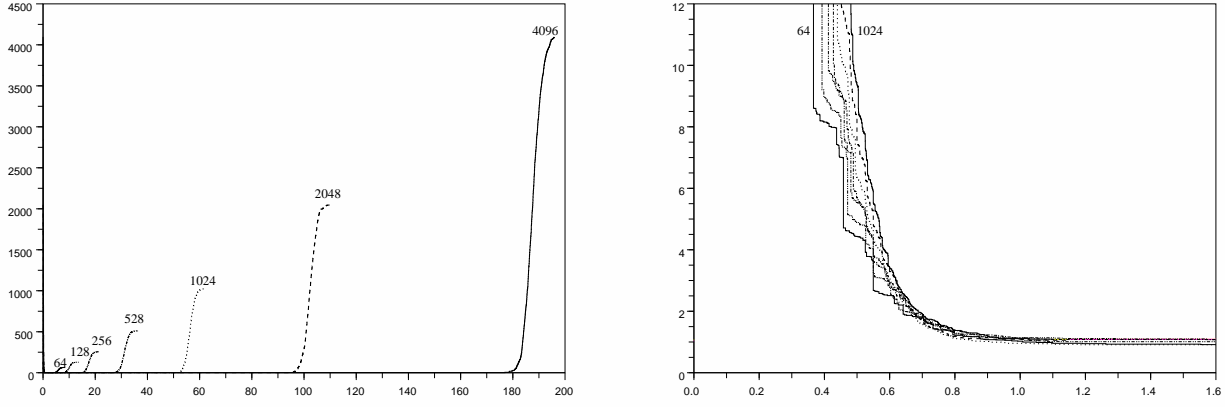


Figure 4: The function  $\gamma \rightarrow \overline{R}_n(\gamma)$  at two scales for 'Haar1' decomposed on the Haar basis and for  $n \in \{64, 128, 256, 512, 1024, 2048, 4096\}$  with  $j_0 = \log_2(n)$ .

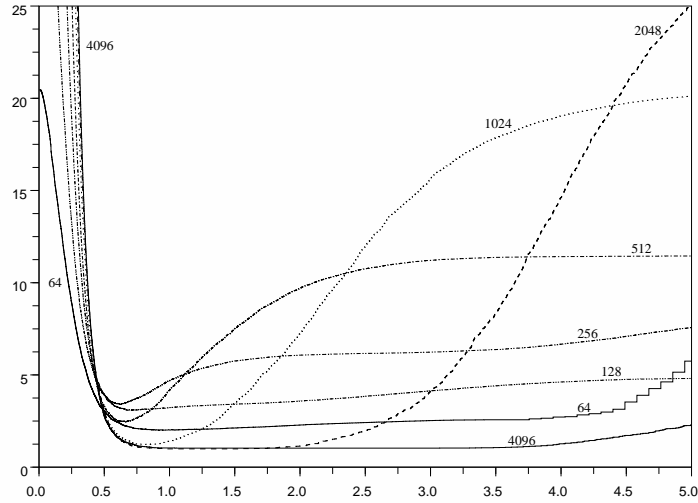


Figure 5: The function  $\gamma \rightarrow \overline{R}_n(\gamma)$  for 'Gauss1' decomposed on the spline basis and for  $n \in \{64, 128, 256, 512, 1024, 2048, 4096\}$  with  $j_0 = \log_2(n)$ .

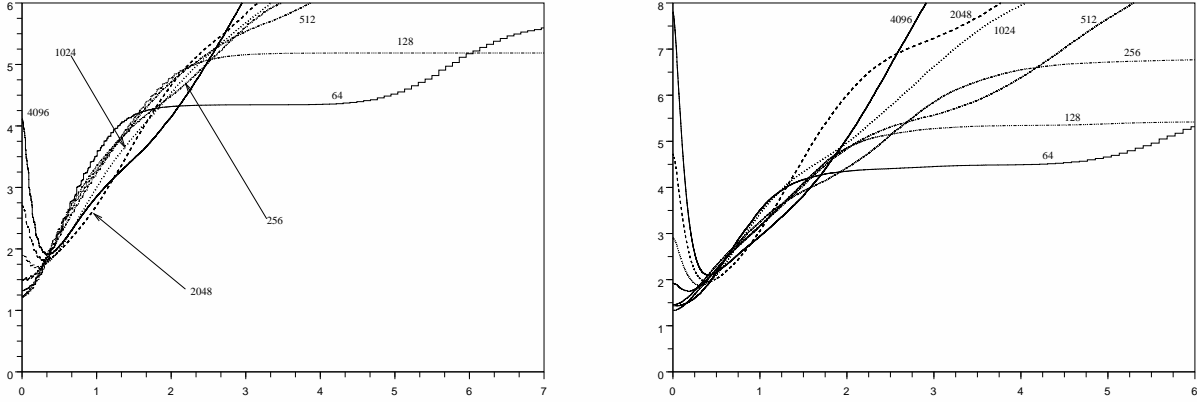


Figure 6: The function  $\gamma \rightarrow \overline{R}_n(\gamma)$  for 'Bumps' decomposed on the Haar and the spline bases and for  $n \in \{64, 128, 256, 512, 1024, 2048, 4096\}$  with  $j_0 = \log_2(n)$ .

To discuss our results, we introduce

$$\gamma_{\min}(n) = \operatorname{argmin}_{\gamma > 0} \overline{R}_n(\gamma).$$

For 'Haar1',  $\gamma_{\min}(n) \geq 1$  for any value of  $n$  and taking  $\gamma < 1$  deteriorates the performances of the estimate. Such a result was established from the theoretical point of view in Theorem 8. In fact, Figure 4 allows to draw the following major conclusion for 'Haar1':

$$\overline{R}_n(\gamma) \approx \overline{R}_n(\gamma_{\min}) \approx 1 \quad (6.1)$$

for a wide range of  $\gamma$  around  $\gamma_{\min} > 1$  that contains  $\gamma = 1$ . For instance, when  $n = 4096$ , the minimum of  $\overline{R}_n$ , close to 1, is very flat and the minimizer is surrounded by the "plateau" [1, 177]. So, the values of  $\gamma_{\min}(n)$  should not be considered as sacred. Our thresholding rule with  $\gamma = 1$  performs very well since it achieves the same performance as the oracle estimator.

For 'Gauss1',  $\gamma_{\min}(n) \geq 0.5$  for any value of  $n$ . Moreover, as soon as  $n$  is large enough, the oracle ratio at  $\gamma_{\min}$  is of order 1. Besides, when  $n \geq 2048$ , as for 'Haar1',  $\gamma_{\min}(n)$  is larger than 1. We observe the "plateau phenomenon" as well and as for 'Haar1', the size of the plateau increases when  $n$  increases. This can be explained by the following important property of 'Gauss1'. 'Gauss1' can be well approximated by a finite combination of the atoms of the spline basis. So, we have the strong impression that the asymptotic result of Theorem 8 could be generalized for the spline basis as soon as we can build positive signals decomposed on the spline basis.

Conclusions for 'Bumps' are very different. Remark that this irregular signal has many significant wavelet coefficients at high resolution levels whatever the basis. We have  $\gamma_{\min}(n) < 0.5$  for each value of  $n$ . Besides,  $\gamma_{\min}(n) \approx 0$  when  $n \leq 256$ , meaning that all the coefficients until  $j = j_0$  have to be kept to obtain the best estimate. So, the parameter  $j_0$  plays an essential role and has to be well calibrated to ensure that there are no non-negligible wavelet coefficients for  $j > j_0$ . Other differences between Figure 4 (or Figure 5) and Figure 6 have to be emphasized. For 'Bumps', when  $n \geq 512$ , the minimum of  $\overline{R}_n$  is well localized, there is no plateau anymore and  $\overline{R}_n(1) > 2 (\overline{R}_n(\gamma_{\min}(n)))$  (larger than 1).

As a preliminary conclusion, it seems that the ideal choice of  $\gamma$  and the performance of the thresholding rule highly depend on the decomposition of the signal on the wavelet basis. Hence, in

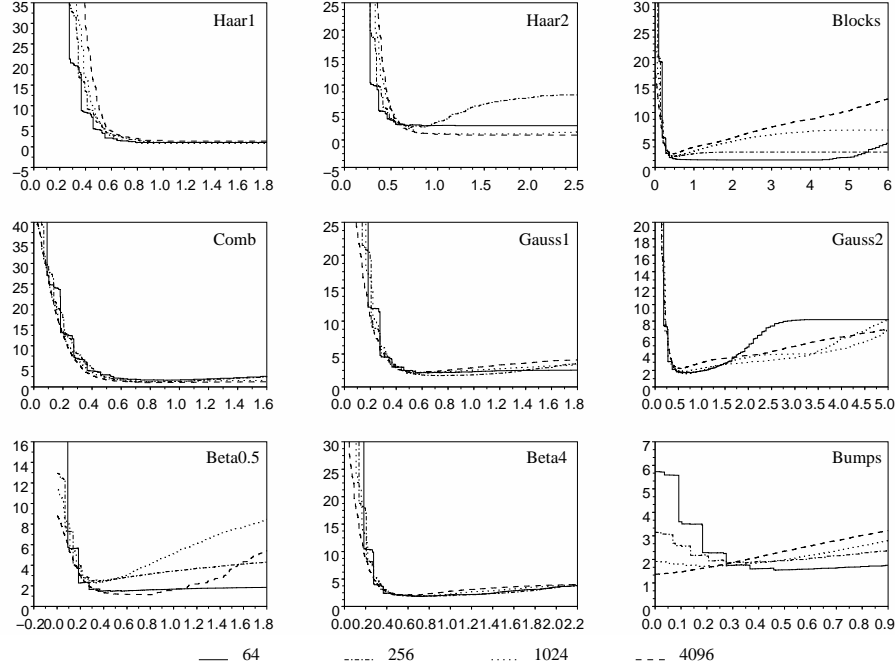


Figure 7: Average over 100 iterations of the function  $R_n$  for signals decomposed on the Haar basis and for  $n \in \{64, 256, 1024, 4096\}$  with  $j_0 = 10$ .

the sequel, we have decided to force  $j_0 = 10$  so that the decomposition on the basis is not too rough. To extend previous results and for the sake of exhaustiveness Figures 7 and 8 display the average of the function  $R_n$  for the signals 'Haar1', 'Haar2', 'Blocks', 'Comb', 'Gauss1', 'Gauss2', 'Beta0.5', 'Beta4' and 'Bumps' with  $j_0 = 10$ . For brevity, we only consider the values  $n \in \{64, 256, 1024, 4096\}$  and the average of  $R_n$  is performed over 100 simulations. Note also that we fix  $j_0 = 10$  and 100 simulations (and not larger parameters) because computational difficulties arise when we deal with infinite support for heavy-tailed signals ('Beta4' and 'Comb') and for a wide range of  $\gamma$ . Figure 7 gives the results obtained for the Haar basis and Figure 8 for the spline basis. To interpret the results, we introduce

$$R_n^{\log}(\gamma) = \frac{\sum_{\lambda \in \Gamma_n} (\tilde{\beta}_\lambda - \beta_\lambda)^2}{\sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n} \log(n))} = \frac{\sum_{\lambda \in \Gamma_n} (\hat{\beta}_\lambda 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} - \beta_\lambda)^2}{\sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n} \log(n))},$$

where the denominator appears in the upper bound of Theorem 2. We also measure the  $\ell_2$ -performance of the estimator by using

$$r_n(\gamma) = \sum_{\lambda \in \Gamma_n} (\tilde{\beta}_\lambda - \beta_\lambda)^2 = \sum_{\lambda \in \Gamma_n} (\hat{\beta}_\lambda 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} - \beta_\lambda)^2.$$

Table 1 gives, for each signal and for  $n \in \{64, 256, 2048, 4096\}$ , the average of  $r_n(1)$ , denoted  $\overline{r_n}(1)$ , the average of  $R_n(1)$ , denoted  $\overline{R_n}(1)$  and the average of  $R_n^{\log}(1)$ , denoted  $\overline{R_n^{\log}}(1)$  (100 simulations are performed). In view of Table 1, let us introduce two classes of functions. The first class is the class of signals that are well approximated by a finite combination of the atoms of the basis (it contains 'Haar1', 'Haar2' and 'Comb' for the Haar basis and 'Gauss1' and 'Gauss2' for the spline

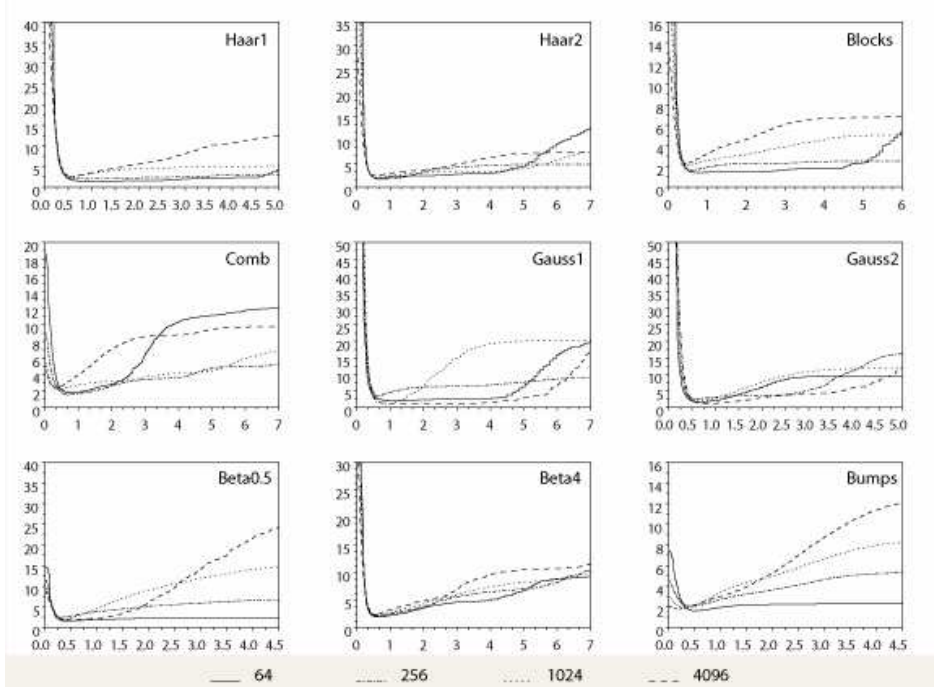


Figure 8: Average over 100 iterations of the function  $R_n$  for signals decomposed on the spline basis and for  $n \in \{64, 256, 1024, 4096\}$  with  $j_0 = 10$ .

basis). For such signals, the estimation problem is close to a parametric problem and in this case the performance of the oracle estimate can be achieved at least for  $n$  large enough and (6.1) is true for a wide range of  $\gamma$  around  $\gamma_{\min}$  that contains  $\gamma = 1$ . The second class is the class of irregular signals with significant wavelet coefficients at high resolution levels (it contains all the other cases except 'Beta0.5'). For such signals, Table 1 shows that  $\overline{R_n}(1)$  seems to increase with  $n$ . But  $R_n^{\log}(1)$  remains constant, showing that the upper bound (with the logarithmic term) of Theorem 2 is probably achieved up to a constant. 'Beta0.5' has only one significant coefficient at each level. This may explain why its behavior seems to be between the first and second class behavior. Finally let us note that the oracle ratio curve for 'Bumps',  $j_0 = 10$  and  $n = 4096$  has a minimizer  $\gamma_{\min}$  close to 0 and has a different behavior from the one with  $j_0 = 12$  (see Figure 6). It illustrates again the fact that 'Bumps' has still some important coefficients at the level of resolution  $j_0 = 12$  that can be taken into account if  $\log_2(n) = 12$ .

Finally, we would like to emphasize the following conclusions. Performances of our thresholding rule are suitable since the ratio  $\overline{R_n}(1)$  is controlled. Moreover a convenient choice of the basis improves this ratio but also the performances of the estimator itself. Furthermore, the size of the support does not play any role (compare estimation of 'Comb' and 'Haar1' for instance) and the estimate  $\tilde{f}_{n,1}$  performs well for recovering the size and location of peaks.

## 7 Proofs

In this section, the notation  $\square$  represents an absolute constant whose value may change at each line. For any  $x > 0$ , the notation  $\lceil x \rceil$  denotes the smallest integer larger than  $x$ . Notations of Sections 2



	$n$	Haar			Spline		
		$\overline{r_n}(1)$	$\overline{R_n}(1)$	$\overline{R_n^{\log}}(1)$	$\overline{r_n}(1)$	$\overline{R_n}(1)$	$\overline{R_n^{\log}}(1)$
Haar1	64	0.016	1.0	0.2	0.10	1.4	0.7
	256	0.0042	1.1	0.2	0.068	2.0	0.8
	1024	0.0008	0.8	0.1	0.042	3.3	0.9
	4096	0.0002	1.0	0.2	0.016	3.5	0.7
Haar2	64	0.082	2.6	0.6	0.21	2.1	1.0
	256	0.026	3.3	0.6	0.085	1.8	0.7
	1024	0.0023	1.2	0.2	0.053	2.4	0.9
	4096	0.0004	1.0	0.1	0.026	2.9	0.8
Blocks	64	0.31	1.4	0.9	0.27	1.4	0.9
	256	0.26	2.5	1.0	0.21	1.9	1.0
	1024	0.13	2.9	0.9	0.13	2.6	0.9
	4096	0.053	3.7	0.8	0.063	3.2	0.8
Comb	64	0.61	1.7	0.4	1.71	1.8	0.8
	256	0.12	1.3	0.2	0.78	1.7	0.7
	1024	0.032	1.4	0.2	0.52	2.7	0.8
	4096	0.0063	1.1	0.1	0.23	4.0	0.7
Gauss1	64	0.21	2.3	0.9	0.10	2.1	0.7
	256	0.072	1.8	0.7	0.060	4.5	0.9
	1024	0.039	2.6	0.7	0.0048	1.2	0.2
	4096	0.018	2.9	0.7	0.0017	1.2	0.2
Gauss2	64	0.17	1.9	0.7	0.12	2.1	0.7
	256	0.07	2.0	0.6	0.05	3.1	0.6
	1024	0.031	2.3	0.6	0.012	2.8	0.4
	4096	0.015	3.0	0.7	0.0017	1.2	0.2
Beta0.5	64	1.6	1.7	1.0	2.2	1.9	1.0
	256	1.1	3.4	1.0	1.4	3.8	1.0
	1024	0.45	5.1	0.8	0.51	4.6	0.8
	4096	0.045	1.6	0.3	0.066	2.3	0.3
Beta4	64	0.25	2.1	0.8	0.36	2.2	0.9
	256	0.093	2.0	0.6	0.16	2.5	0.8
	1024	0.041	2.2	0.6	0.061	2.7	0.7
	4096	0.020	2.8	0.7	0.024	3.3	0.6
Bumps	64	4.9	1.8	1.0	4.3	2.0	1.1
	256	3.1	2.5	1.0	2.5	2.7	1.0
	1024	1.5	3.0	0.9	1.2	3.4	0.9
	4096	0.62	3.4	0.7	0.38	3.0	0.6

Table 1: Values of  $\overline{r_n}(1)$ ,  $\overline{R_n}(1)$  and  $\overline{R_n^{\log}}(1)$  for each signal decomposed on the Haar basis or the spline basis and for  $n \in \{64, 256, 1024, 4096\}$ .

and 3 are used. Recall also that we have set

$$\forall \lambda \in \Lambda, \quad F_\lambda = \int_{\text{supp}(\varphi_\lambda)} f(x) dx.$$

### 7.1 Proof of Theorem 1

Let  $\gamma, p, q, \varepsilon$  be as in Theorem 1. We start as usual for model selection with (1.3). One has for all subset  $m$  of  $\Gamma_n$

$$\gamma_n(\tilde{f}_{n,\gamma}) + \text{pen}(\hat{m}) \leq \gamma_n(\hat{f}_m) + \text{pen}(m).$$

If  $g = \sum_{\lambda \in \Lambda} \alpha_\lambda \tilde{\varphi}_\lambda$ , setting  $\nu_n(g) = \sum_{\lambda \in \Lambda} \alpha_\lambda (\hat{\beta}_\lambda - \beta_\lambda)$ , we obtain that

$$\gamma_n(g) = \|g - f\|_{\tilde{\varphi}}^2 - \|f\|_{\tilde{\varphi}}^2 - 2\nu_n(g).$$

Hence,

$$\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2 \leq \|\hat{f}_m - f\|_{\tilde{\varphi}}^2 + 2\nu_n(\tilde{f}_{n,\gamma} - \hat{f}_m) + \text{pen}(m) - \text{pen}(\hat{m}).$$

For any subset of indices  $m'$ , let  $\chi(m') = \sqrt{\sum_{\lambda \in m'} (\hat{\beta}_\lambda - \beta_\lambda)^2}$  and let  $f_m = \sum_{\lambda \in m} \beta_\lambda \tilde{\varphi}_\lambda$  be the orthogonal projection of  $f$  on  $S_m$  for  $\|\cdot\|_{\tilde{\varphi}}$ . Then  $\chi^2(m) = \nu_n(\hat{f}_m - f_m) = \|\hat{f}_m - f_m\|_{\tilde{\varphi}}^2 = \|\hat{f}_m - f\|_{\tilde{\varphi}}^2 - \|f_m - f\|_{\tilde{\varphi}}^2$ . Hence,

$$\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2 \leq \|f_m - f\|_{\tilde{\varphi}}^2 - \chi^2(m) + 2\nu_n(\tilde{f}_{n,\gamma} - f_m) + \text{pen}(m) - \text{pen}(\hat{m}).$$

Furthermore,

$$\nu_n(\tilde{f}_{n,\gamma} - f_m) \leq \|\tilde{f}_{n,\gamma} - f_m\|_{\tilde{\varphi}} \chi(m \cup \hat{m}) \leq \|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}} \chi(m \cup \hat{m}) + \|f_m - f\|_{\tilde{\varphi}} \chi(m \cup \hat{m}).$$

Using twice the fact that  $2ab \leq \theta a^2 + \theta^{-1}b^2$ , for  $\theta = 2/(2 + \varepsilon)$  and  $\theta = 2/\varepsilon$ , we obtain that

$$2\nu_n(\tilde{f}_{n,\gamma} - f_m) \leq \frac{2}{2 + \varepsilon} \|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2 + \frac{2}{\varepsilon} \|f_m - f\|_{\tilde{\varphi}}^2 + (1 + \varepsilon) \chi^2(m \cup \hat{m}).$$

Hence we obtain that

$$\frac{\varepsilon}{2 + \varepsilon} \|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2 \leq \left(1 + \frac{2}{\varepsilon}\right) \sum_{\lambda \notin m} \beta_\lambda^2 + (1 + \varepsilon) \chi^2(m \cup \hat{m}) - \chi^2(m) + \text{pen}(m) - \text{pen}(\hat{m}).$$

But  $\chi^2(m \cup \hat{m}) \leq \chi^2(m) + \chi^2(\hat{m})$ . After integration it remains to control

$$\mathcal{A} = \mathbb{E}((1 + \varepsilon) \chi^2(\hat{m}) - \text{pen}(\hat{m})).$$

Since

$$\hat{m} = \left\{ \lambda \in \Gamma_n : |\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma} \right\},$$

we have

$$\mathcal{A} = \sum_{\lambda \in \Gamma_n} \mathbb{E} \left( \left[ (1 + \varepsilon) (\hat{\beta}_\lambda - \beta_\lambda)^2 - \eta_{\lambda,\gamma}^2 \right] 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} \right).$$

Hence,

$$\mathcal{A} \leq \sum_{\lambda \in \Gamma_n} \mathbb{E} \left( (1 + \varepsilon) (\hat{\beta}_\lambda - \beta_\lambda)^2 1_{(1 + \varepsilon) (\hat{\beta}_\lambda - \beta_\lambda)^2 \geq \eta_{\lambda,\gamma}^2} 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} \right).$$

Then, remark that if  $|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}$  then  $|\hat{\beta}_\lambda| \geq \frac{\mu \log n}{n} \|\varphi_\lambda\|_\infty$ , where  $\mu = [\sqrt{6} + 1/3]\gamma$  but also that  $|\hat{\beta}_\lambda| \leq \frac{\|\varphi_\lambda\|_\infty N_\lambda}{n}$ , hence  $N_\lambda \geq \mu \log n$ , where

$$N_\lambda = \int_{\text{supp}(\varphi_\lambda)} dN.$$

So, one can split  $\mathcal{A}$  and bound this term by  $LDLM + LDSM$ , where

$$LDLM = \sum_{\lambda \in \Gamma_n} \mathbb{E} \left( (1 + \varepsilon) (\hat{\beta}_\lambda - \beta_\lambda)^2 1_{(1+\varepsilon)(\hat{\beta}_\lambda - \beta_\lambda)^2 \geq \eta_{\lambda,\gamma}^2} 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} 1_{N_\lambda \geq \mu \log n} 1_{nF_\lambda \geq \theta \mu \log n} \right),$$

and

$$LDSM = \sum_{\lambda \in \Gamma_n} \mathbb{E} \left( (1 + \varepsilon) (\hat{\beta}_\lambda - \beta_\lambda)^2 1_{(1+\varepsilon)(\hat{\beta}_\lambda - \beta_\lambda)^2 \geq \eta_{\lambda,\gamma}^2} 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} 1_{N_\lambda \geq \mu \log n} 1_{nF_\lambda \leq \theta \mu \log n} \right),$$

where  $\theta < 1$  is a parameter that is chosen later on. Here,  $LDLM$  stands for “large deviation large mass” and  $LDSM$  stands for “large deviation small mass”. Let us begin with  $LDLM$ . By the Hölder Inequality

$$LDLM \leq \sum_{\lambda \in \Gamma_n} (1 + \varepsilon) [\mathbb{E} |\hat{\beta}_\lambda - \beta_\lambda|^{2p}]^{1/p} \mathbb{P}(|\hat{\beta}_\lambda - \beta_\lambda| \geq \eta_{\lambda,\gamma} / \sqrt{1 + \varepsilon})^{1/q} 1_{nF_\lambda \geq \theta \mu \log n}.$$

Before going further, let us state the following useful lemma:

**Lemma 1.** *For any  $u > 0$*

$$\mathbb{P} \left( |\hat{\beta}_\lambda - \beta_\lambda| \geq \sqrt{2uV_{\lambda,n}} + \frac{\|\varphi_\lambda\|_\infty u}{3n} \right) \leq 2e^{-u}. \quad (7.1)$$

Moreover, for any  $u > 0$

$$\mathbb{P} \left( V_{\lambda,n} \geq \tilde{V}_{\lambda,n}(u) \right) \leq e^{-u}, \quad (7.2)$$

where

$$\tilde{V}_{\lambda,n}(u) = \hat{V}_{\lambda,n} + \sqrt{2\hat{V}_{\lambda,n} \frac{\|\varphi_\lambda\|_\infty^2}{n^2} u} + 3 \frac{\|\varphi_\lambda\|_\infty^2}{n^2} u.$$

**Proof.** Equation (7.1) easily comes from the classical inequalities (see Kingman’s book [26] or Equation (5.2) of [31]). The same classical inequalities applied to  $-\varphi_\lambda^2/n^2$  instead of  $\varphi_\lambda/n$  give that

$$\mathbb{P} \left( V_{\lambda,n} \geq \hat{V}_{\lambda,n} + \sqrt{2u \int_{\mathbb{R}} \frac{\varphi_\lambda^4(x)}{n^4} n f(x) dx} + \frac{\|\varphi_\lambda\|_\infty^2}{3n^2} u \right) \leq e^{-u}.$$

But one can remark that

$$\int_{\mathbb{R}} \frac{\varphi_\lambda^4(x)}{n^4} n f(x) dx \leq \frac{\|\varphi_\lambda\|_\infty^2}{n^2} V_{\lambda,n}.$$

Set  $a = u \frac{\|\varphi_\lambda\|_\infty^2}{n^2}$ , then

$$\mathbb{P}(V_{\lambda,n} - \sqrt{2V_{\lambda,n}a} - a/3 \geq \hat{V}_{\lambda,n}) \leq e^{-u}.$$

Let  $\mathcal{P}(x) = x^2 - \sqrt{2a}x - a/3$ . The discriminant of this polynomial is  $10a/3$  which is strictly larger than  $2a$ . Since  $V_{\lambda,n}$  and  $\hat{V}_{\lambda,n}$  are positive, this means that one can inverse the equation  $\mathcal{P}(\sqrt{V_{\lambda,n}}) = \hat{V}_{\lambda,n}$  and we obtain

$$\mathbb{P}(\sqrt{V_{\lambda,n}} \geq \mathcal{P}^{-1}(\hat{V}_{\lambda,n})) \leq e^{-u}.$$

But  $\mathcal{P}^{-1}(\hat{V}_{\lambda,n})$  is the positive solution of

$$(\mathcal{P}^{-1}(\hat{V}_{\lambda,n}))^2 - \sqrt{2a}\mathcal{P}^{-1}(\hat{V}_{\lambda,n}) - (a/3 + \hat{V}_{\lambda,n}) = 0.$$

So, finally,  $\mathcal{P}^{-1}(\hat{V}_{\lambda,n}) = \sqrt{\hat{V}_{\lambda,n} + 5a/6} + \sqrt{a/2}$ . To conclude it remains to remark that  $\tilde{V}_{\lambda,n} \geq (\mathcal{P}^{-1}(\hat{V}_{\lambda,n}))^2$ .  $\blacksquare$

Using Equations (7.1) and (7.2) of Lemma 1, we have

$$\begin{aligned} & \mathbb{P}(|\hat{\beta}_\lambda - \beta_\lambda| \geq \eta_{\lambda,\gamma}/\sqrt{1+\varepsilon}) \\ & \leq \mathbb{P}\left(|\hat{\beta}_\lambda - \beta_\lambda| \geq \sqrt{\frac{2\gamma \log n}{1+\varepsilon} \tilde{V}_{\lambda,n}(\gamma \log n)} + \frac{\gamma \log n \|\varphi_\lambda\|_\infty}{3(1+\varepsilon)n}\right) \\ & \leq \mathbb{P}\left(|\hat{\beta}_\lambda - \beta_\lambda| \geq \sqrt{\frac{2\gamma \log n}{1+\varepsilon} \tilde{V}_{\lambda,n}(\gamma \log n)} + \frac{\gamma \log n \|\varphi_\lambda\|_\infty}{3(1+\varepsilon)n}, V_{\lambda,n} \geq \tilde{V}_{\lambda,n}(\gamma \log n)\right) \\ & + \mathbb{P}\left(|\hat{\beta}_\lambda - \beta_\lambda| \geq \sqrt{\frac{2\gamma \log n}{1+\varepsilon} \tilde{V}_{\lambda,n}(\gamma \log n)} + \frac{\gamma \log n \|\varphi_\lambda\|_\infty}{3(1+\varepsilon)n}, V_{\lambda,n} < \tilde{V}_{\lambda,n}(\gamma \log n)\right) \\ & \leq \mathbb{P}(V_{\lambda,n} \geq \tilde{V}_{\lambda,n}(\gamma \log n)) + \mathbb{P}\left(|\hat{\beta}_\lambda - \beta_\lambda| \geq \sqrt{\frac{2\gamma}{1+\varepsilon} \log n V_{\lambda,n}} + \frac{\gamma \log n \|\varphi_\lambda\|_\infty}{3(1+\varepsilon)n}\right) \\ & \leq n^{-\gamma} + 2n^{-\gamma/(1+\varepsilon)} \\ & \leq 3n^{-\gamma/(1+\varepsilon)}. \end{aligned}$$

We need another lemma which looks like the Rosenthal inequality.

**Lemma 2.** *For all  $p \geq 2$ , there exists some absolute constant  $C$  such that*

$$\mathbb{E}(|\hat{\beta}_\lambda - \beta_\lambda|^{2p}) \leq C^p p^{2p} \left( V_{\lambda,n}^p + \left[ \frac{\|\varphi_\lambda\|_\infty}{n} \right]^{2p-2} V_{\lambda,n} \right).$$

**Proof.** We know that a Poisson process is infinitely divisible. This means that for all positive integer  $k$  one can see  $N$  as the reunion of  $k$  iid Poisson processes,  $N^i$  with intensity (here)  $nk^{-1} \times f$  with respect to the Lebesgue measure. Hence, one can apply Rosenthal inequalities for all  $k$ , saying that

$$\hat{\beta}_\lambda - \beta_\lambda = \sum_{i=1}^k \int \frac{\varphi_\lambda(x)}{n} (dN_x^i - nk^{-1}f(x)dx) = \sum_{i=1}^k Y_i$$

where for any  $i$ ,

$$Y_i = \int \frac{\varphi_\lambda(x)}{n} (dN_x^i - nk^{-1}f(x)dx).$$

So the  $Y_i$ 's are iid centered variables, all having a moment of order  $2p$ . We apply Rosenthal's inequality (see Theorem 2.5 of [22]) on the positive and negative parts of  $Y_i$ . This easily implies that

$$\mathbb{E} \left( \left| \sum_{i=1}^k Y_i \right|^{2p} \right) \leq K(p) \max \left( \left( \mathbb{E} \sum_{i=1}^k Y_i^2 \right)^p, \left( \mathbb{E} \sum_{i=1}^k |Y_i|^{2p} \right) \right),$$

where

$$K(p) \leq \left( 8 \times \frac{2p}{\log(2p)} \right)^{2p}.$$

It remains to bound the upper limit of  $\mathbb{E}(\sum_{i=1}^k |Y_i|^q)$  for all  $q \in \{2p, 2\} \geq 2$  when  $k \rightarrow \infty$ . Let us introduce

$$\Omega_k = \{\forall i \in \{1, \dots, k\}, N_{\mathbb{R}}^i \leq 1\}.$$

Then, it is easy to see that  $\mathbb{P}(\Omega_k^c) \leq k^{-1}(n\|f\|_1)^2$  (see e.g., (7.5) below).

On  $\Omega_k$ ,  $|Y_i|^q = O_k(k^{-q})$  if  $\int \frac{\varphi_{\lambda}(x)}{n} dN_x^i = 0$  and  $|Y_i|^q = \left[ \frac{|\varphi_{\lambda}(T)|}{n} \right]^q + O_k \left( k^{-1} \left[ \frac{|\varphi_{\lambda}(T)|}{n} \right]^{q-1} \right)$  if  $\int \frac{\varphi_{\lambda}(x)}{n} dN_x^i = \frac{\varphi_{\lambda}(T)}{n}$  where  $T$  is the point of the process  $N^i$ . Consequently,

$$\begin{aligned} \mathbb{E} \sum_{i=1}^k |Y_i|^q &\leq \mathbb{E} \left( 1_{\Omega_k} \left( \sum_{T \in N} \left[ \left[ \frac{|\varphi_{\lambda}(T)|}{n} \right]^q + O_k \left( k^{-1} \left[ \frac{|\varphi_{\lambda}(T)|}{n} \right]^{q-1} \right) \right] + k O_k(k^{-q}) \right) \right) \\ &\quad + \sqrt{\mathbb{P}(\Omega_k^c)} \sqrt{\mathbb{E} \left[ \left( \sum_{i=1}^k |Y_i|^q \right)^2 \right]}. \end{aligned} \quad (7.3)$$

But,

$$\begin{aligned} \sum_{i=1}^k |Y_i|^q &\leq 2^{q-1} \left( \sum_{i=1}^k \left[ \left[ \frac{\|\varphi_{\lambda}\|_{\infty}}{n} \right]^q (N_{\mathbb{R}}^i)^q + \left( k^{-1} \int |\varphi_{\lambda}(x)| f(x) dx \right)^q \right] \right) \\ &\leq 2^{q-1} \left( \left[ \frac{\|\varphi_{\lambda}\|_{\infty}}{n} \right]^q N_{\mathbb{R}}^q + k \left( k^{-1} \int |\varphi_{\lambda}(x)| f(x) dx \right)^q \right). \end{aligned}$$

So, when  $k \rightarrow +\infty$ , the last term in (7.3) converges to 0 since a Poisson variable has moments of every order and

$$\limsup_{k \rightarrow \infty} \mathbb{E} \sum_{i=1}^k |Y_i|^q \leq \mathbb{E} \left( \int \left[ \frac{|\varphi_{\lambda}(x)|}{n} \right]^q dN_x \right) \leq \left[ \frac{\|\varphi_{\lambda}\|_{\infty}}{n} \right]^{q-2} V_{\lambda, n},$$

which concludes the proof. ■

Since

$$\left[ \frac{\|\varphi_{\lambda}\|_{\infty}}{n} \right]^{2p-2} V_{\lambda, n} \leq \max \left( V_{\lambda, n}^p, \left[ \frac{\|\varphi_{\lambda}\|_{\infty}}{n} \right]^{2p} \right),$$

there exists some constant  $\tilde{C}$  such that

$$\mathbb{E}(|\hat{\beta}_{\lambda} - \beta_{\lambda}|^{2p}) \leq \tilde{C}^p p^{2p} \left( V_{\lambda, n}^p + \left[ \frac{\|\varphi_{\lambda}\|_{\infty}}{n} \right]^{2p} \right).$$

Finally,

$$LDLM \leq \square(1 + \varepsilon)p^2 n^{-\gamma/(q(1+\varepsilon))} \sum_{\lambda \in \Gamma_n} \left( V_{\lambda,n} + \left( \frac{\|\varphi_\lambda\|_\infty}{n} \right)^2 \right) 1_{nF_\lambda \geq \theta \mu \log n}.$$

Since  $\|\varphi_\lambda\|_\infty \leq c_{\varphi,n} \sqrt{n}$  for all  $\lambda \in \Gamma_n$ , one has

$$\begin{aligned} LDLM &\leq \square(1 + \varepsilon)p^2 c_{\varphi,n}^2 n^{-\gamma/(q(1+\varepsilon))} \sum_{\lambda \in \Gamma_n} \left( F_\lambda + \frac{1}{n} \right) 1_{nF_\lambda \geq \theta \mu \log n} \\ &\leq \square(1 + \varepsilon)p^2 c_{\varphi,n}^2 n^{-\gamma/(q(1+\varepsilon))} \left( \sum_{\lambda \in \Gamma_n} F_\lambda + \frac{1}{n} \sum_{\lambda \in \Gamma_n} \frac{nF_\lambda}{\theta \mu \log n} \right). \end{aligned}$$

But,

$$\sum_{\lambda \in \Gamma_n} F_\lambda = \sum_{\lambda \in \Gamma_n} \int f(x) 1_{x \in \text{supp}(\varphi_\lambda)} dx = \int f(x) dx \sum_{\lambda \in \Gamma_n} 1_{x \in \text{supp}(\varphi_\lambda)}. \quad (7.4)$$

Using (2.1), we then have

$$\sum_{\lambda \in \Gamma_n} F_\lambda \leq \|f\|_1 m_{\varphi,n} \log n.$$

This is exactly what we need for the first part provided that  $\theta$  is an absolute constant and  $\mu > 1$ . Now we go back to  $LDSM$ . Applying the Hölder inequality again one obtains,

$$LDSM \leq (1 + \varepsilon) \sum_{\lambda \in \Gamma_n} \mathbb{E}(|\hat{\beta}_\lambda - \beta_\lambda|^{2p})^{1/p} \mathbb{P}(N_\lambda - nF_\lambda \geq (1 - \theta)\mu \log n)^{1/q}.$$

To deal with this term, we state the following result.

**Lemma 3.** *There exists an absolute constant  $0 < \theta < 1$  such that if  $nF_\lambda \leq \theta \mu \log n$ , then, for all  $n$  such that  $(1 - \theta)\mu \log n \geq 2$ ,*

$$\mathbb{P}(N_\lambda - nF_\lambda \geq (1 - \theta)\mu \log n) \leq F_\lambda n^{-\gamma}.$$

**Proof.** We use the same classical inequalities (see Kingman's book [26] or equation (5.2) of [31]).

$$\mathbb{P}(N_\lambda - nF_\lambda \geq (1 - \theta)\mu \log n) \leq \exp \left( - \frac{((1 - \theta)\mu \log n)^2}{2(nF_\lambda + (1 - \theta)\mu \log n/3)} \right) \leq n^{-\frac{3(1-\theta)^2}{2(2\theta+1)}\mu}.$$

If  $nF_\lambda \geq n^{-\gamma-1}$ , then provided that  $\frac{3(1-\theta)^2}{2(2\theta+1)}\mu \geq 2\gamma + 2$ , one has the result. This imposes the value of  $\theta$ . Indeed since

$$\frac{3(1-\theta)^2}{2(2\theta+1)}\mu = \frac{3(1-\theta)^2}{2(2\theta+1)}(\sqrt{6} + 1/3)\gamma$$

one takes  $\theta$  such that

$$\frac{3(1-\theta)^2}{2(2\theta+1)}(\sqrt{6} + 1/3) = 4.$$

If  $nF_\lambda \leq n^{-\gamma-1}$ ,

$$\begin{aligned} \mathbb{P}(N_\lambda - nF_\lambda \geq (1 - \theta)\mu \log n) &\leq \mathbb{P}(N_\lambda > (1 - \theta)\mu \log n) \leq \mathbb{P}(N_\lambda \geq 2) \\ &\leq \sum_{k \geq 2} \frac{(nF_\lambda)^k}{k!} e^{-nF_\lambda} \leq (nF_\lambda)^2 \leq F_\lambda n^{-\gamma}. \end{aligned} \quad (7.5)$$

■

We apply Lemma 3 to bound the deviation and Lemma 2 to bound  $\mathbb{E}(|\hat{\beta}_\lambda - \beta_\lambda|^{2p})$ . Hence,

$$LDSM \leq \square(1 + \varepsilon)p^2 n^{-\gamma/q} \sum_{\lambda \in \Gamma_n} \left( V_{\lambda,n} + \left[ \frac{\|\varphi_\lambda\|_\infty}{n} \right]^{2-2/p} V_{\lambda,n}^{1/p} \right) F_\lambda^{1/q}.$$

Since  $\|\varphi_\lambda\|_\infty \leq c_{\varphi,n}\sqrt{n}$ ,

$$LDSM \leq \square(1 + \varepsilon)p^2 c_{\varphi,n}^2 n^{-\gamma/q} \sum_{\lambda \in \Gamma_n} (F_\lambda^{1+1/q} + F_\lambda).$$

Finally, as previously, by using (7.4)

$$LDSM \leq \square(1 + \varepsilon)p^2 c_{\varphi,n}^2 m_{\varphi,n} n^{-\gamma/q} \log(n) (\|f\|_1) \max(\|f\|_1, 1)^{1/q}.$$

## 7.2 Proof of Theorem 2

At first, we apply Theorem 1 with  $c_{\varphi,n} = \|\varphi\|_\infty 2^{j_0/2} n^{-1/2}$ . For the last term, we want to prove that one can always find  $q$  and  $\varepsilon$  such that  $2^{j_0} n^{-\gamma/(q(1+\varepsilon)) - 1} \log(n) = o(n^{-1})$ . But if  $\gamma > c$  then one can always find  $q > 1$  and  $\varepsilon > 0$  such that  $\gamma > cq(1 + \varepsilon)$  and this implies also that  $\gamma > 1 + \varepsilon$ . So, by exchanging the infimum and the expectation we obtain that

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_\varphi^2) \leq (1 + 2\varepsilon^{-1}) \inf_{m \subset \Gamma_n} \left\{ (1 + 2\varepsilon^{-1}) \sum_{\lambda \notin m} \beta_\lambda^2 + \sum_{\lambda \in m} [\varepsilon V_{\lambda,n} + \mathbb{E}(\eta_{\lambda,\gamma}^2)] \right\} + \frac{C_2(\gamma, \|f\|_1, c, c', \varphi)}{n}.$$

But for all  $\delta > 0$ ,

$$\mathbb{E}(\eta_{\lambda,\gamma}^2) \leq (1 + \delta) 2\gamma \log n \mathbb{E}(\tilde{V}_{\lambda,n}) + (1 + \delta^{-1}) \left( \frac{\gamma \log n}{3n} \right)^2 \|\varphi_\lambda\|_\infty^2.$$

Moreover

$$\mathbb{E}(\tilde{V}_{\lambda,n}) \leq (1 + \delta) V_{\lambda,n} + (1 + \delta^{-1}) 3\gamma \log n \frac{\|\varphi_\lambda\|_\infty^2}{n^2}.$$

So, finally for all  $\delta > 0$ ,

$$\begin{aligned} \mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_\varphi^2) &\leq (1 + 2\varepsilon^{-1}) \\ \inf_{m \subset \Gamma_n} &\left\{ (1 + 2\varepsilon^{-1}) \sum_{\lambda \notin m} \beta_\lambda^2 + \sum_{\lambda \in m} [\varepsilon + (1 + \delta)^2 2\gamma \log n] V_{\lambda,n} + c(\delta, \gamma) \sum_{\lambda \in m} \left( \frac{\log n \|\varphi_\lambda\|_\infty}{n} \right)^2 \right\} \\ &+ \frac{C_2(\gamma, \|f\|_1, c, c', \varphi)}{n}, \quad (7.6) \end{aligned}$$

where  $c(\delta, \gamma)$  is a positive constant. One needs the following lemma.

**Lemma 4.** *We set*

$$S_\varphi = \max \left\{ \sup_{x \in \text{supp}(\phi)} |\phi(x)|, \sup_{x \in \text{supp}(\psi)} |\psi(x)| \right\}$$

and

$$I_\varphi = \min\left\{\inf_{x \in \text{supp}(\phi)} |\phi(x)|, \inf_{x \in \text{supp}(\psi)} |\psi(x)|\right\}.$$

Using (3.2), we define  $\Theta_\varphi = \frac{S_\varphi^2}{I_\varphi^2}$ . We have, for all  $\lambda \in \Lambda$ ,

$$\text{- if } F_\lambda \leq \Theta_\varphi \frac{\log(n)}{n}, \text{ then } \beta_\lambda^2 \leq \Theta_\varphi^2 \sigma_\lambda^2 \frac{\log(n)}{n},$$

$$\text{- if } F_\lambda > \Theta_\varphi \frac{\log(n)}{n}, \text{ then } \|\varphi_\lambda\|_\infty \frac{\log(n)}{n} \leq \sigma_\lambda \sqrt{\frac{\log(n)}{n}}.$$

**Proof.** We note  $\lambda = (j, k)$  and assume that  $j \geq 0$  (arguments are similar for  $j = -1$ ).

If  $F_\lambda \leq \Theta_\varphi \frac{\log(n)}{n}$ , we have

$$\begin{aligned} |\beta_\lambda| &\leq S_\psi 2^{j/2} F_\lambda \\ &\leq S_\varphi 2^{j/2} \sqrt{F_\lambda} \sqrt{\Theta_\varphi} \sqrt{\frac{\log(n)}{n}} \\ &\leq S_\varphi I_\varphi^{-1} \sqrt{\Theta_\varphi} \sigma_\lambda \sqrt{\frac{\log(n)}{n}} \\ &\leq \Theta_\varphi \sigma_\lambda \sqrt{\frac{\log(n)}{n}}, \end{aligned}$$

since

$$\sigma_\lambda^2 \geq I_\varphi^2 2^j F_\lambda.$$

For the second point, observe that

$$\sigma_\lambda \sqrt{\frac{\log(n)}{n}} \geq 2^{j/2} I_\varphi \sqrt{\Theta_\varphi} \frac{\log(n)}{n}$$

and

$$\|\psi_\lambda\|_\infty \frac{\log(n)}{n} \leq 2^{j/2} S_\varphi \frac{\log(n)}{n}.$$

■

Now let us apply (7.6) for some fixed  $\delta, \varepsilon$  to

$$m = \left\{ \lambda \in \Gamma_n : \beta_\lambda^2 > \Theta_\varphi^2 \frac{\sigma_\lambda^2}{n} \log n \right\}.$$



This implies that for all  $\lambda \in m$ ,  $F_\lambda > \Theta_\varphi \frac{\log(n)}{n}$ . So, since  $\Theta_\varphi \geq 1$ ,

$$\begin{aligned}
\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_\varphi^2) &\leq C(\gamma) \times \\
&\left[ \sum_{\lambda \in \Gamma_n} \beta_\lambda^2 1_{\beta_\lambda^2 \leq \Theta_\varphi^2 \frac{\sigma_\lambda^2}{n} \log n} + \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 + \sum_{\lambda \in \Gamma_n} \left[ \frac{\log n}{n} \sigma_\lambda^2 + \left( \frac{\log n}{n} \right)^2 \|\varphi_\lambda\|_\infty^2 \right] 1_{\beta_\lambda^2 > \Theta_\varphi^2 \frac{\sigma_\lambda^2}{n} \log n, F_\lambda > \Theta_\varphi \frac{\log(n)}{n}} \right] \\
&\quad + \frac{C_2(\gamma, \|f\|_1, c, c', \varphi)}{n} \\
&\leq C(\gamma) \left[ \sum_{\lambda \in \Gamma_n} \left( \beta_\lambda^2 1_{\beta_\lambda^2 \leq \Theta_\varphi^2 V_{\lambda,n} \log n} + 2 \log n V_{\lambda,n} 1_{\beta_\lambda^2 > \Theta_\varphi^2 V_{\lambda,n} \log n} \right) + \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \right] + \\
&\quad + \frac{C_2(\gamma, \|f\|_1, c, c', \varphi)}{n} \\
&\leq C_1(\gamma) \left[ \sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, \Theta_\varphi^2 V_{\lambda,n} \log n) + \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \right] + \frac{C_2(\gamma, \|f\|_1, c, c', \varphi)}{n},
\end{aligned}$$

where  $C(\gamma)$  and  $C_1(\gamma)$  are positive quantities depending only on  $\gamma$ .

### 7.3 Proof of Theorem 3

Let us assume that  $f$  belongs to  $B_{2,\Gamma}^{\frac{\alpha}{1+2\alpha}}(R^{\frac{1}{1+2\alpha}}) \cap W_\alpha(R) \cap \mathbb{L}_1(R) \cap \mathbb{L}_2(R)$ . Inequality (4.1) of Theorem 2 implies for all  $n$ ,

$$\begin{aligned}
\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_\varphi^2) &\leq C_1(\gamma, \varphi) \left[ \sum_{\lambda \in \Gamma_n} \left( \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda \sqrt{\frac{\log n}{n}}} + V_{\lambda,n} \log n 1_{|\beta_\lambda| > \sigma_\lambda \sqrt{\frac{\log n}{n}}} \right) + \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \right] + \\
&\quad + \frac{C_2(\gamma, R, c, c', \varphi)}{n}.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{\lambda \in \Gamma_n} V_{\lambda,n} \log n 1_{|\beta_\lambda| > \sigma_\lambda \sqrt{\frac{\log n}{n}}} &= \sum_{\lambda \in \Gamma_n} \sigma_\lambda^2 \frac{\log n}{n} \sum_{k=0}^{+\infty} 1_{2^{-k-1} \beta_\lambda^2 \leq \sigma_\lambda^2 \frac{\log n}{n} < 2^{-k} \beta_\lambda^2} \\
&\leq \sum_{k=0}^{+\infty} 2^{-k} \sum_{\lambda \in \Lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq 2^{(k+1)/2} \sigma_\lambda \sqrt{\frac{\log n}{n}}} \\
&\leq \sum_{k=0}^{+\infty} 2^{-k} R^{\frac{2}{1+2\alpha}} \left( 2^{(k+1)/2} \sqrt{\frac{\log n}{n}} \right)^{\frac{4\alpha}{1+2\alpha}} \\
&\leq R^{\frac{2}{1+2\alpha}} \rho_{n,\alpha}^2 \sum_{k=0}^{+\infty} 2^{-k + \frac{2\alpha(k+1)}{1+2\alpha}}
\end{aligned}$$

and

$$\sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \leq R^{\frac{2}{1+2\alpha}} \rho_{n,\alpha}^2.$$

So,

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \leq C(\gamma, \varphi, \alpha) R^{\frac{2}{1+2\alpha}} \rho_{n,\alpha}^2 + \frac{C_2(\gamma, R, c, c', \varphi)}{n},$$

where  $C(\gamma, \varphi, \alpha)$  depends on  $\gamma$ , the basis and  $\alpha$ . Hence,

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \leq C(\gamma, \varphi, \alpha) R^{\frac{2}{1+2\alpha}} \rho_{n,\alpha}^2 (1 + o_n(1))$$

and  $f$  belongs to  $MS(\tilde{f}_\gamma, \rho_\alpha)(R')$  for  $R'$  large enough.

Conversely, let us suppose that  $f$  belongs to  $MS(\tilde{f}_\gamma, \rho_\alpha)(R') \cap \mathbb{L}_1(R') \cap \mathbb{L}_2(R')$ . Then, for any  $n$ ,

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \leq R'^2 \left( \frac{\log n}{n} \right)^{\frac{2\alpha}{1+2\alpha}}.$$

Consequently, for any  $n$ ,

$$\sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \leq R'^2 \left( \frac{\log n}{n} \right)^{\frac{2\alpha}{1+2\alpha}}.$$

This implies that  $f$  belongs to  $B_{2,\Gamma}^{\frac{\alpha}{1+2\alpha}}(R')$ .

Now, we want to prove that  $f \in W_\alpha(R)$  for  $R > 0$ . We have

$$\sum_{\lambda \in \Lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda \sqrt{\frac{\gamma \log n}{2n}}} \leq \sum_{\lambda \in \Gamma_n} \beta_\lambda^2 + \sum_{\lambda \in \Gamma_n} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda \sqrt{\frac{\gamma \log n}{2n}}}.$$

But  $\tilde{\beta}_\lambda = \hat{\beta}_\lambda 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}}$ , so,

$$|\beta_\lambda| 1_{|\beta_\lambda| \leq \frac{\eta_{\lambda,\gamma}}{2}} \leq |\beta_\lambda - \tilde{\beta}_\lambda|.$$

So, for any  $n$ ,

$$\begin{aligned} \sum_{\lambda \in \Lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda \sqrt{\frac{\gamma \log n}{2n}}} &\leq \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 + \mathbb{E} \left\{ \sum_{\lambda \in \Gamma_n} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda \sqrt{\frac{\gamma \log n}{2n}}} [1_{|\beta_\lambda| \leq \frac{\eta_{\lambda,\gamma}}{2}} + 1_{|\beta_\lambda| > \frac{\eta_{\lambda,\gamma}}{2}}] \right\} \\ &\leq \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 + \sum_{\lambda \in \Gamma_n} \mathbb{E}[(\tilde{\beta}_\lambda - \beta_\lambda)^2] + \sum_{\lambda \in \Gamma_n} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda \sqrt{\frac{\gamma \log n}{2n}}} \mathbb{E}(1_{|\beta_\lambda| > \frac{\eta_{\lambda,\gamma}}{2}}) \\ &\leq \sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 + \sum_{\lambda \in \Gamma_n} \mathbb{E}[(\tilde{\beta}_\lambda - \beta_\lambda)^2] + \sum_{\lambda \in \Gamma_n} \beta_\lambda^2 \mathbb{P} \left( \sigma_\lambda \sqrt{\frac{\gamma \log n}{2n}} > \frac{\eta_{\lambda,\gamma}}{2} \right) \\ &\leq \mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) + \sum_{\lambda \in \Gamma_n} \beta_\lambda^2 \mathbb{P} \left( \sigma_\lambda \sqrt{\frac{\gamma \log n}{2n}} > \frac{\eta_{\lambda,\gamma}}{2} \right). \end{aligned}$$

Using Lemma 1,

$$\mathbb{P} \left( \sigma_\lambda \sqrt{\frac{2\gamma \log n}{n}} > \eta_{\lambda,\gamma} \right) \leq \mathbb{P}(\tilde{V}_{\lambda,n} \leq V_{\lambda,n}) \leq n^{-\gamma}$$

and

$$\sum_{\lambda \in \Lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda \sqrt{\frac{\gamma \log n}{2n}}} \leq (R')^2 \left( \sqrt{\frac{\log n}{n}} \right)^{\frac{4\alpha}{1+2\alpha}} + \|f\|_{\tilde{\varphi}}^2 n^{-\gamma}.$$

Since this is true for every  $n$ , we have for any  $t \leq 1$ ,

$$\sum_{\lambda \in \Lambda} \beta_{\lambda}^2 1_{|\beta_{\lambda}| \leq \sigma_{\lambda} t} \leq R^{\frac{2}{1+2\alpha}} \left( \sqrt{\frac{2}{\gamma}} t \right)^{\frac{4\alpha}{1+2\alpha}}, \quad (7.7)$$

where  $R$  is a constant large enough depending on  $R'$ . Note that

$$\sup_{t \geq 1} t^{\frac{-4\alpha}{1+2\alpha}} \sum_{\lambda \in \Lambda} \beta_{\lambda}^2 1_{|\beta_{\lambda}| \leq \sigma_{\lambda} t} \leq \|f\|_{\tilde{\varphi}}^2.$$

We conclude that

$$f \in B_{2,\Gamma}^{\frac{\alpha}{1+2\alpha}}(R) \cap W_{\alpha}(R)$$

for  $R$  large enough.

## 7.4 Proof of Proposition 1

Since  $\beta < 1/2$ ,  $f_{\beta} \in \mathbb{L}_1 \cap \mathbb{L}_2$ . If the Haar basis is considered, the wavelet coefficients  $\beta_{j,k}$  of  $f_{\beta}$  can be calculated and we obtain for any  $j \geq 0$ , for any  $k \notin \{0, \dots, 2^j - 1\}$ ,  $\beta_{j,k} = 0$  and for any  $j \geq 0$ , for any  $k \in \{0, \dots, 2^j - 1\}$ ,

$$\beta_{j,k} = (1 - \beta)^{-1} 2^{-j(\frac{1}{2}-\beta)} \left( 2 \left( k + \frac{1}{2} \right)^{1-\beta} - k^{1-\beta} - (k+1)^{1-\beta} \right)$$

and there exists a constant  $0 < c_{1,\beta} < \infty$  only depending on  $\beta$  such that

$$\lim_{k \rightarrow \infty} 2^{j(\frac{1}{2}-\beta)} k^{1+\beta} \beta_{j,k} = c_{1,\beta}.$$

Moreover the  $\beta_{j,k}$ 's are strictly positive. Consequently they can be bounded up and below, up to a constant, by  $2^{-j(\frac{1}{2}-\beta)} k^{-(1+\beta)}$ . Similarly, for any  $j \geq 0$ , for any  $k \in \{0, \dots, 2^j - 1\}$ ,

$$\sigma_{j,k}^2 = (1 - \beta)^{-1} 2^{j\beta} \left( (k+1)^{1-\beta} - k^{1-\beta} \right).$$

and there exists a constant  $0 < c_{2,\beta} < \infty$  only depending on  $\beta$  such that

$$\lim_{k \rightarrow \infty} 2^{-j\beta} k^{\beta} \sigma_{j,k}^2 = c_{2,\beta}.$$

There exist two constants  $\kappa(\beta)$  and  $\kappa'(\beta)$  only depending on  $\beta$  such that for any  $0 < t < 1$ ,

$$|\beta_{j,k}| \leq t \sigma_{j,k} \Rightarrow k \geq \kappa(\beta) t^{-\frac{2}{\beta+2}} 2^{j\left(\frac{\beta-1}{\beta+2}\right)}$$

and

$$\kappa(\beta) t^{-\frac{2}{\beta+2}} 2^{j\left(\frac{\beta-1}{\beta+2}\right)} \geq 2^j \iff 2^j \leq \kappa'(\beta) t^{-\frac{2}{3}}.$$

So, if  $2^j \leq \kappa'(\beta) t^{-\frac{2}{3}}$ , since  $\beta_{j,k} = 0$  for  $k \geq 2^j$ ,

$$\sum_{k \in \mathbb{Z}} \beta_{j,k}^2 1_{\beta_{j,k} \leq t \sigma_{j,k}} = 0.$$

We obtain

$$\sum_{\lambda \in \Lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq t\sigma_\lambda} \leq C(\beta) \sum_{j=-1}^{+\infty} 2^{-j(1-2\beta)} 1_{2^j > \kappa'(\beta)t^{-\frac{2}{3}}} \sum_{k=1}^{2^j-1} k^{-2-2\beta} \leq C'(\beta) t^{\frac{2-4\beta}{3}},$$

where  $C(\beta)$  and  $C'(\beta)$  denote two constants only depending on  $\beta$ . So, for any  $0 < \alpha < \frac{1}{4}$ , if we take  $\beta \leq \frac{1-4\alpha}{2+4\alpha}$ , then, for any  $0 < t < 1$ ,  $t^{\frac{2-4\beta}{3}} \leq t^{\frac{4\alpha}{1+2\alpha}}$ . Finally, there exists  $c \geq 1$ , such that for any  $n$ ,

$$\sum_{\lambda \notin \Gamma_n} \beta_\lambda^2 \leq R^2 \rho_{n,\alpha}^2$$

where  $R > 0$ . And in this case,

$$f_\beta \notin \mathbb{L}_\infty, \quad f_\beta \in \mathcal{B}_{2,\infty}^{\frac{\alpha}{c(1+2\alpha)}} \cap W_\alpha := MS(\tilde{f}_\gamma, \rho_\alpha).$$

## 7.5 Proof of Theorem 4

Since

$$\forall \lambda = (j, k), \quad \sigma_\lambda^2 \leq \min [\max(2^j; 1) \|\varphi\|_\infty^2 F_{j,k} ; \|f\|_\infty \|\varphi\|_2^2], \quad (7.8)$$

where  $\varphi \in \{\phi, \psi\}$  according to the value of  $j$ , we have for any  $t > 0$  and any  $\tilde{J} \geq 0$

$$\begin{aligned} \sum_{\lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda t} &\leq \sum_{j < \tilde{J}} \sum_k \sigma_{j,k}^2 t^2 + \sum_{j \geq \tilde{J}} \sum_k \beta_{j,k}^2 \left( \frac{\sigma_{j,k} t}{|\beta_{j,k}|} \right)^{2-p} \\ &\leq \max(\|\phi\|_\infty^2, \|\psi\|_\infty^2) t^2 \sum_{j < \tilde{J}} 2^j \sum_k F_{j,k} + \sum_{j \geq \tilde{J}} \sum_k \beta_{j,k}^2 \left( \frac{t \sqrt{\|f\|_\infty \|\psi\|_2^2}}{|\beta_{j,k}|} \right)^{2-p} \\ &\leq c(\varphi, R') \left( 2^{\tilde{J}} t^2 + t^{2-p} \sum_{j \geq \tilde{J}} \sum_k |\beta_{j,k}|^p \right), \end{aligned}$$

where  $c(\varphi, R')$  is a constant only depending on the basis and on  $R'$ . Now, let us assume that  $f$  belongs to  $\mathcal{B}_{p,\infty}^\alpha(R)$  (that contains  $\mathcal{B}_{p,q}^\alpha(R)$ , see Section 3), with  $\alpha + \frac{1}{2} - \frac{1}{p} > 0$ . Then,

$$\sum_{\lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda t} \leq c_1(\varphi, \alpha, p, R') \left( 2^{\tilde{J}} t^2 + t^{2-p} R^p 2^{-\tilde{J}p(\alpha + \frac{1}{2} - \frac{1}{p})} \right).$$

where  $c_1(\varphi, \alpha, p, R')$  depends on the basis,  $\alpha$ ,  $p$  and  $R'$ . With  $\tilde{J}$  such that

$$2^{\tilde{J}} \leq R^{\frac{2}{1+2\alpha}} t^{\frac{-2}{1+2\alpha}} < 2^{\tilde{J}+1},$$

$$\sum_{\lambda} \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda t} \leq c_2(\varphi, \alpha, p, R') R^{\frac{2}{1+2\alpha}} t^{\frac{4\alpha}{1+2\alpha}}$$

where  $c_2(\varphi, \alpha, p, R')$  depends on the basis,  $\alpha$ ,  $p$  and  $R'$ . So,  $f$  belongs to  $W_\alpha(R'')$  for  $R''$  large enough.

Furthermore, using (3.3), if  $p \leq 2$  and

$$\alpha \left( 1 - \frac{1}{c(1+2\alpha)} \right) \geq \frac{1}{p} - \frac{1}{2}$$

$$\mathcal{B}_{p,\infty}^\alpha(R) \subset \overline{\mathcal{B}_{2,\infty}^{\frac{\alpha}{c(1+2\alpha)}}}(R).$$

Finally, for  $R''$  large enough,

$$\mathcal{B}_{p,q}^\alpha(R) \subset \mathcal{B}_{p,\infty}^\alpha(R) \subset \overline{\mathcal{B}_{2,\infty}^{\frac{\alpha}{c(1+2\alpha)}}}(R'') \cap W_\alpha(R'').$$

We recall

$$MS(\tilde{f}_\gamma, \rho_\alpha) := \overline{\mathcal{B}_{2,\infty}^{\frac{\alpha}{c(1+2\alpha)}}} \cap W_\alpha,$$

which proves (4.3).

Moreover

$$\inf_{\tilde{f}} \sup_{f \in \mathcal{B}_{p,q}^\alpha(R) \cap \mathcal{L}_{1,2,\infty}(R')} \mathbb{E}(\|\tilde{f} - f\|^2) \geq C(\alpha, R, R') n^{-\frac{2\alpha}{2\alpha+1}},$$

where  $C(\alpha, R, R')$  is a constant. Indeed, using computations similar to those of Theorem 2 of [17], it is easy to prove that if  $K$  is a compact interval and  $\mathcal{B}_{p,q,K}^\alpha(R)$  is the set of functions supported by  $K$  and belonging to  $\mathcal{B}_{p,q}^\alpha(R)$  the minimax risk associated with  $\mathcal{B}_{p,q,K}^\alpha(R)$  is larger than  $n^{-2\alpha/(1+2\alpha)}$  up to a constant.

But (4.4) implies that  $\alpha > \alpha^*$  and  $p > p^*$  satisfy (4.2). This proves the adaptive minimax properties of  $\tilde{f}_\gamma$  stated in the theorem.

## 7.6 Proof of Theorem 5

The proof is established for  $p < \infty$ . Similar arguments lead to the same results for  $p = \infty$ . Let us fix real numbers  $n_* > 1$  and  $f_* > 1$  and let us define the following increasing sequence

$$a_0 = 0, \quad a_1 = 4 \quad \text{and} \quad \forall l \geq 1, \quad a_{l+1} = 2a_l + 2^{\lceil n_* l \rceil + 1}.$$

Let  $b_l = \frac{a_{l+1}}{2} - 1$ . Let  $I_{j,k}^+ = [k2^{-j}, (k+1/2)2^{-j}]$  and  $I_{j,k}^- = [(k+1/2)2^{-j}, (k+1)2^{-j}]$ . Set for all  $x \in \mathbb{R}$ ,

$$f_l(x) = \sum_{m=a_l}^{b_l} 2^{(1-f_*)l+1} 1_{I_{l,m}^+}$$

and

$$f(x) = \sum_{l=0}^{+\infty} f_l(x).$$

The  $f_l$ 's have support in  $S_l = [a_l 2^{-l}, a_{l+1} 2^{-(l+1)}]$ . All the  $S_l$ 's are disjoint and we can prove by an easy induction that all the  $a_l 2^{-l}$ 's are even positive integer numbers (indeed,  $a_{l+1} 2^{-(l+1)} = 2^{\lceil n_* l \rceil - l} + a_l 2^{-l}$  and  $\lceil n_* l \rceil - l > 0$  if  $l \neq 0$ ).

Now, let us compute the wavelet coefficients associated with  $f$  denoted  $\beta_{j,k}$  for  $j \geq 0$  and for any  $k \in \mathbb{Z}$  and  $\alpha_k = \beta_{-1,k}$  for any  $k \in \mathbb{Z}$ . We are working with the Haar basis. Recall that the spaces considered are viewed as sequence spaces.

For the  $\beta_{j,k}$ 's, let us remark that  $\text{supp}(\varphi_{j,k})$  is always included between two successive integers, consequently there exists a unique  $l_{j,k}$  such that  $\text{supp}(\varphi_{j,k}) \subset S_{l_{j,k}}$ . So,

$$\beta_{j,k} = \int f_{l_{j,k}} \varphi_{j,k}.$$

Moreover, if  $j \neq l_{j,k}$ , the coefficient is zero: either  $j > l_{j,k}$  and  $\varphi_{j,k}$  sees only one flat line, or  $j < l_{j,k}$  and  $\varphi_{j,k}$  integrates the same number of flat pieces in  $I_{j,k}^+$  and  $I_{j,k}^-$ ; since the pieces have all the same level, this is also 0. Finally, for  $j = l_{j,k}$ , the computation is easy and we find

$$\beta_{j,k} = 2^{(1-f_*)j+1} \int_{I_{j,k}^+} 2^{j/2} [1_{I_{j,k}^+} - 1_{I_{j,k}^-}] \times 1_{a_j \leq k \leq b_j} = 2^{-j(f_*-1/2)} 1_{a_j \leq k \leq b_j}.$$

For the coefficients  $\alpha_k$ 's, there exists also a unique  $l_k$  such that  $\text{supp}(\varphi_{-1,k}) \subset S_{l_k}$  and

$$\alpha_k = 2^{(1-f_*)l_k+1} \frac{1}{2} = \sum_l 2^{(1-f_*)l} 1_{a_l 2^{-l} \leq k < a_{l+1} 2^{-(l+1)}}.$$

Now, we want to compute  $\sigma_{j,k}$  when  $\beta_{j,k} \neq 0$ . If  $j \geq 0$

$$\begin{aligned} F_{j,k} &= \int_{\text{supp}(\psi_{j,k})} f(x) dx = 2^{j(1-f_*)} 2^{-j} = 2^{-jf_*}, \\ \sigma_{j,k}^2 &= \int \psi_{j,k}^2(x) f(x) dx = 2^j \int_{\text{supp}(\psi_{j,k})} f(x) dx = 2^j F_{j,k} = 2^{j(1-f_*)}. \end{aligned}$$

If  $j = -1$

$$\sigma_{j,k} = F_{j,k} = \alpha_k.$$

Now, we fix the parameter  $n_*$  and  $f_*$  such that

1.  $\|f\|_1 < \infty$ ,  $\|f\|_2 < \infty$ ,  $\|f\|_\infty < \infty$ ,
2.  $f \in \mathcal{B}_{p,\infty}^\alpha$ ,
3.  $f \notin W_\alpha$ .

Since  $f_* > 1$ , then  $\|f\|_\infty < \infty$ . We have

$$\|f\|_1 = \sum_{l=0}^{+\infty} \sum_{m=a_l}^{b_l} 2^{(1-f_*)l+1} 2^{-l-1} = \sum_{l=0}^{+\infty} 2^{\lceil n_* l \rceil} 2^{-f_* l} < \infty \iff f_* > n_*. \quad (7.9)$$

We have for all  $j \geq 0$

$$\begin{aligned} \sum_k |\beta_{j,k}|^p &= \sum_{k=a_j}^{b_j} |2^{-j(f_*-1/2)}|^p \\ &= 2^{\lceil n_* j \rceil} 2^{jp/2} 2^{-jf_* p}. \end{aligned}$$

Then,

$$\begin{aligned} f \in \mathcal{B}_{p,\infty}^\alpha &\iff \exists R > 0, \forall j \geq 0, 2^{j(n_*+p/2-f_*p)} \leq R^p 2^{-jp(\alpha+1/2-1/p)} \\ &\iff n_* + p/2 - f_*p \leq -p\alpha - p/2 + 1 \\ &\iff n_* \leq pf_* - p + 1 - p\alpha. \end{aligned} \quad (7.10)$$

Indeed, note that we have

$$\sum_{k \in \mathbb{Z}} |\alpha_k|^p = \sum_{l \geq 0} 2^{\lceil n_* l \rceil - l} \left( 2^{(1-f_*)l} \right)^p < \infty$$

if and only if  $n_* - 1 + p - f_* p < 0$ , which is true as soon as  $f_* > n_*$ . Note also that

$$\|f\|_2 < \infty \iff 2f_* > 1 + n_*,$$

which is also true as soon as  $f_* > n_*$ .

Now, we would like to build  $f$  such that  $f$  does not belong to  $W_\alpha$ . We have for any  $t < 1$ ,

$$\begin{aligned} \sum_{k=a_j}^{b_j} \beta_{j,k}^2 1_{|\beta_{j,k}| \leq t\sigma_{j,k}} &= \sum_{k=a_j}^{b_j} 2^{-2j(f_*-1/2)} 1_{2^{-j(f_*-1/2)} \leq t 2^{j(1-f_*)/2}} \\ &= 2^{j(1-2f_*)} 2^{\lceil n_* j \rceil} 1_{2^{-j f_*} \leq t^2}. \end{aligned}$$

So, with  $j = \lceil \log_2(t^{-2/f_*}) \rceil$ ,

$$\begin{aligned} \sup_{t < 1} t^{-4\alpha/(1+2\alpha)} \sum_j \sum_{k=a_j}^{b_j} \beta_{j,k}^2 1_{|\beta_{j,k}| \leq t\sigma_{j,k}} &= +\infty \iff \sup_{t < 1} t^{-4\alpha/(1+2\alpha)} t^{-2(1+n_*-2f_*)/f_*} = +\infty \\ &\iff -2(1+n_*-2f_*)/f_* < 4\alpha/(1+2\alpha) \\ &\iff 2f_* - n_* - 1 < \frac{2\alpha f_*}{1+2\alpha} \\ &\iff n_* > -1 + \frac{2f_*(1+\alpha)}{1+2\alpha}, \end{aligned} \quad (7.11)$$

and in this case,  $f \notin W_\alpha$ . Now, we choose  $n_* > 1$  and  $f_* > 1$  such that (7.9), (7.10) and (7.11) are satisfied. For this purpose, we take

$$f_* = 1 + 2\alpha - \delta \in \left] (1+2\alpha) \frac{(p\alpha + p - 2)}{2p\alpha + p - 2\alpha - 2}, 1 + 2\alpha \right[$$

for  $\delta \in ]0, \alpha[$  and  $\delta$  small enough. Note that  $p > 2$  implies

$$(1+2\alpha) \frac{(p\alpha + p - 2)}{2p\alpha + p - 2\alpha - 2} < 1 + 2\alpha.$$

We also take

$$n_* = \min(f_* - \delta', pf_* - p + 1 - p\alpha) \in ]1, pf_* - p + 1 - p\alpha]$$

for  $\delta'$  small enough. Note that

$$pf_* - p + 1 - p\alpha = p(1 + 2\alpha - \delta) - p + 1 - p\alpha = p\alpha + 1 - p\delta > 1.$$

With such a choice, we have  $n_* < f_*$  and  $n_* \leq pf_* - p + 1 - p\alpha$ . So (7.9) and (7.10) are satisfied. It remains to check (7.11). We have

$$\begin{aligned} pf_* - p + 1 - p\alpha > -1 + \frac{2f_*(1+\alpha)}{1+2\alpha} &\iff f_* \left[ \frac{2(1+\alpha)}{1+2\alpha} - p \right] < 2 - p - p\alpha \\ &\iff f_*(2 + 2\alpha - p - 2p\alpha) < (1+2\alpha)(2 - p - p\alpha) \\ &\iff f_* > (1+2\alpha) \frac{(p\alpha + p - 2)}{2p\alpha + p - 2\alpha - 2}, \end{aligned}$$

and

$$\begin{aligned}
f_* - \delta' > -1 + \frac{2f_*(1+\alpha)}{1+2\alpha} &\iff f_* \left[ \frac{2(1+\alpha)}{1+2\alpha} - 1 \right] < 1 - \delta' \\
&\iff 2(1+\alpha)f_* - f_*(1+2\alpha) < (1+2\alpha)(1-\delta') \\
&\iff f_* < (1+2\alpha)(1-\delta'),
\end{aligned}$$

which is true for  $\delta'$  small enough. So (7.11) is satisfied, which concludes the proof of the theorem.

## 7.7 Proof of Theorem 6

The proof is established for  $q = \infty$  and  $p < \infty$ . Similar arguments lead to the same results for  $p = \infty$ . In the sequel,  $C$  designates a constant depending on  $R'$ ,  $\gamma$ ,  $c$ ,  $c'$ , on the parameters of the Besov ball, on the basis and that may change at each line. We have for any  $0 < t < 1$  and any  $j \geq 0$ ,

$$\sum_k \beta_{j,k}^2 1_{|\beta_{j,k}| \leq t\sigma_{j,k}} \leq \left( \sum_k |\beta_{j,k}|^p \right)^{\frac{1}{p}} \left( \sum_k |\beta_{j,k}|^r 1_{|\beta_{j,k}| \leq t\sigma_{j,k}} \right)^{\frac{1}{r}} \quad (7.12)$$

with  $\frac{1}{p} + \frac{1}{r} = 1$ . So, using (7.8), we have if  $f \in \mathbb{L}_\infty(R') \cap \mathbb{L}_1(R') \cap \mathcal{B}_{p,\infty}^\alpha(R)$ ,

$$\begin{aligned}
\sum_k \beta_{j,k}^2 1_{|\beta_{j,k}| \leq t\sigma_{j,k}} &\leq C 2^{-j(\alpha + \frac{1}{2} - \frac{1}{p})} \left( \sum_k |\beta_{j,k}| (t\sigma_{j,k})^{r-1} \right)^{\frac{1}{r}} \\
&\leq C 2^{-j(\alpha + \frac{1}{2} - \frac{1}{p})} \left( \sum_k |\beta_{j,k}| t^{r-1} \right)^{\frac{1}{r}} \\
&\leq C 2^{-j(\alpha + \frac{1}{2} - \frac{1}{p} - \frac{1}{2r})} t^{1 - \frac{1}{r}}.
\end{aligned}$$

Indeed,

$$f \in \mathbb{L}_1(R') \Rightarrow \sum_k |\beta_{j,k}| \leq C 2^{\frac{j}{2}} \quad (7.13)$$

(see [24], p. 197). So, for  $\alpha > 1/(2p)$ , we have for any  $t > 0$  and any  $\tilde{J} \geq 0$

$$\begin{aligned}
\sum_\lambda \beta_\lambda^2 1_{|\beta_\lambda| \leq \sigma_\lambda t} &= \sum_j \sum_k \beta_{j,k}^2 1_{|\beta_{j,k}| \leq \sigma_{j,k} t} \\
&\leq C \left[ t^2 \sum_{j < \tilde{J}} 2^j \sum_k F_{j,k} + \sum_{j \geq \tilde{J}} 2^{-j(\alpha + \frac{1}{2} - \frac{1}{p} - \frac{1}{2r})} t^{1 - \frac{1}{r}} \right] \text{ using (7.8) again} \\
&\leq C \left[ t^2 2^{\tilde{J}} + 2^{-\tilde{J}(\alpha - \frac{1}{2p})} t^{1 - \frac{1}{r}} \right].
\end{aligned}$$

With

$$2^{\tilde{J}} \leq t^{-\frac{1 + \frac{1}{p}}{\alpha + \frac{1}{2} + \frac{1}{2r}}} < 2^{\tilde{J}+1}$$



we have

$$\sum_{\lambda} \beta_{\lambda}^2 1_{|\beta_{\lambda}| \leq \sigma_{\lambda} t} \leq C t^{\frac{2\alpha}{\alpha + \frac{1}{2} + \frac{1}{2p}}}.$$

We obtain

$$\sum_{\lambda} \beta_{\lambda}^2 1_{|\beta_{\lambda}| \leq \sigma_{\lambda} t} \leq C t^{\frac{2\alpha}{\alpha + 1 - \frac{1}{2p}}}.$$

So, with  $t = \sqrt{\frac{\log n}{n}}$ ,

$$\sum_{\lambda \in \Gamma_n} \beta_{\lambda}^2 1_{|\beta_{\lambda}| \leq \sigma_{\lambda} \sqrt{\frac{\log n}{n}}} \leq C \left( \frac{\log n}{n} \right)^{\frac{\alpha}{\alpha + 1 - \frac{1}{2p}}}.$$

Furthermore,

$$\begin{aligned} \sum_{\lambda \in \Gamma_n} V_{\lambda, n} \log n 1_{|\beta_{\lambda}| > \sigma_{\lambda} \sqrt{\frac{\log n}{n}}} &= \sum_{\lambda \in \Gamma_n} \sigma_{\lambda}^2 \frac{\log n}{n} \sum_{k=0}^{+\infty} 1_{2^{-k-1} \beta_{\lambda}^2 \leq \sigma_{\lambda}^2 \frac{\log n}{n} < 2^{-k} \beta_{\lambda}^2} \\ &\leq \sum_{k=0}^{+\infty} 2^{-k} \sum_{\lambda \in \Lambda} \beta_{\lambda}^2 1_{|\beta_{\lambda}| \leq 2^{(k+1)/2} \sigma_{\lambda} \sqrt{\frac{\log n}{n}}} \\ &\leq C \sum_{k=0}^{+\infty} 2^{-k} \left( 2^{(k+1)/2} \sqrt{\frac{\log n}{n}} \right)^{\frac{2\alpha}{\alpha + 1 - \frac{1}{2p}}} \\ &\leq C \left( \frac{\log n}{n} \right)^{\frac{\alpha}{\alpha + 1 - \frac{1}{2p}}} \sum_{k=0}^{+\infty} 2^{-k + \frac{\alpha(k+1)}{1 + \alpha - \frac{1}{2p}}} \\ &\leq C \left( \frac{\log n}{n} \right)^{\frac{\alpha}{\alpha + 1 - \frac{1}{2p}}}. \end{aligned}$$

Now, using (7.12), (7.13) and (3.3) we have when  $\lambda = (j, k) \notin \Gamma_n$ ,

$$\begin{aligned} \sum_k \beta_{j,k}^2 &\leq C 2^{-j(\alpha + \frac{1}{2} - \frac{1}{p})} \left( \sum_k |\beta_{j,k}| (\sup_k |\beta_{j,k}|)^{r-1} \right)^{\frac{1}{r}} \\ &\leq C 2^{-j(\alpha + \frac{1}{2} - \frac{1}{p})} \left( \sum_k |\beta_{j,k}| 2^{-\frac{j(r-1)}{2}} \right)^{\frac{1}{r}} \\ &\leq C 2^{-j\alpha}. \end{aligned}$$

and applying Theorem 2, we obtain for  $c \geq 1$ ,

$$\sum_{\lambda \notin \Gamma_n} \beta_{\lambda}^2 \leq C \left( \frac{\log n}{n} \right)^{\frac{\alpha}{\alpha + 1 - \frac{1}{2p}}}$$

and

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \leq C \left( \frac{\log n}{n} \right)^{\frac{\alpha}{\alpha + 1 - \frac{1}{2p}}}.$$

## 7.8 Proof of Theorem 7

Let us consider the Haar basis. For  $j \geq 0$  and  $D \in \{0, 1, \dots, 2^j\}$ , we set

$$\mathcal{C}_{j,D} = \{f_m = \rho 1_{[0,1]} + a_{j,D} \sum_{k \in m} \tilde{\varphi}_{j,k} : |m| = D, m \subset \mathcal{N}_j\},$$

where

$$\mathcal{N}_j = \{k : \tilde{\varphi}_{j,k} \text{ has support in } [0, 1]\}.$$

The parameters  $j, D, \rho, a_{j,D}$  is chosen later to fulfill some requirements. Note that

$$N_j = \text{card}(\mathcal{N}_j) = 2^j.$$

We know that there exists a subset of  $\mathcal{C}_{j,D}$ , denoted  $\mathcal{M}_{j,D}$ , and some universal constants, denoted  $\theta'$  and  $\sigma$ , such that for all  $m, m' \in \mathcal{M}_{j,D}$ ,

$$\text{card}(m \Delta m') \geq \theta' D, \quad \log(\text{card}(\mathcal{M}_{j,D})) \geq \sigma D \log\left(\frac{2^j}{D}\right)$$

(see Lemma 8 of [31]). Now, let us describe all the requirements necessary to obtain the lower bound of the risk.

- To ensure  $f_m \geq 0$  and the equivalence between the Kullback distance and the  $\mathbb{L}_2$ -norm (see below), the  $f_m$ 's have to be larger than  $\rho/2$ . Since the  $\tilde{\varphi}_{j,k}$ 's have disjoint support, this means that

$$\rho \geq 2^{1+j/2} |a_{j,D}|. \quad (7.14)$$

- We need the  $f_m$ 's to be in  $\mathbb{L}_1(R'') \cap \mathbb{L}_\infty(R'')$ . Since  $\|f\|_1 = \rho$  and  $\|f\|_\infty = \rho + 2^{j/2} |a_{j,D}|$ , we need

$$\rho + 2^{j/2} |a_{j,D}| \leq R''. \quad (7.15)$$

- The  $f_m$ 's have to belong to  $\mathcal{B}_{2,\infty}^{\frac{\alpha}{1+2\alpha}}(R')$  i.e.

$$\rho + 2^{j\alpha/(1+2\alpha)} \sqrt{D} |a_{j,D}| \leq R'. \quad (7.16)$$

- The  $f_m$ 's have to belong to  $W_\alpha(R)$ . We have  $\sigma_\lambda^2 = \rho$ . Hence for any  $t > 0$

$$\rho^2 1_{\rho \leq \sqrt{\rho t}} + D a_{j,D}^2 1_{|a_{j,D}| \leq \sqrt{\rho t}} \leq R^{2/(1+2\alpha)} t^{4\alpha/(1+2\alpha)}.$$

If  $|a_{j,D}| \leq \rho$ , then it is enough to have

$$\rho^2 + D a_{j,D}^2 \leq R^{2/(1+2\alpha)} \rho^{2\alpha/(1+2\alpha)} \quad (7.17)$$

and

$$D a_{j,D}^2 \leq R^{2/(1+2\alpha)} \left( \frac{a_{j,D}^2}{\rho} \right)^{2\alpha/(1+2\alpha)}. \quad (7.18)$$

If the parameters satisfy these equations, then

$$\mathcal{R}(W_\alpha(R) \cap \mathcal{B}_{2,\infty}^{\frac{\alpha}{1+2\alpha}}(R') \cap \mathcal{L}_{1,2,\infty}(R'')) \geq \mathcal{R}(\mathcal{M}_{j,D}).$$

Moreover if for any estimator  $\hat{f}$ , we define  $\hat{f}' = \arg \inf_{g \in \mathcal{M}_{j,D}} \|g - \hat{f}\|_{\tilde{\varphi}}$ , then for  $f \in \mathcal{M}_{j,D}$ ,

$$\|f - \hat{f}'\|_{\tilde{\varphi}} \leq \|f - \hat{f}\|_{\tilde{\varphi}} + \|\hat{f} - \hat{f}'\|_{\tilde{\varphi}} \leq 2\|f - \hat{f}\|_{\tilde{\varphi}}.$$

Hence,

$$\mathcal{R}(\mathcal{M}_{j,D}) \geq \frac{1}{4} \inf_{\hat{f} \in \mathcal{M}_{j,D}} \sup_{f \in \mathcal{M}_{j,D}} \mathbb{E}(\|f - \hat{f}\|_{\tilde{\varphi}}^2).$$

But for every  $m \neq m'$ ,  $\|f_m - f_{m'}\|_{\tilde{\varphi}}^2 = \sum_{k \in m \Delta m'} a_{j,D}^2 \geq \theta' D a_{j,D}^2$ . Hence,

$$\mathcal{R}(\mathcal{M}_{j,D}) \geq \frac{1}{4} \theta' D a_{j,D}^2 \inf_{\hat{f} \in \mathcal{M}_{j,D}} (1 - \inf_{f \in \mathcal{M}_{j,D}} \mathbb{P}(\hat{f} = f)).$$

We now use Fano's Lemma of [5], and to do so we need to provide an upper bound of the Kullback-Leibler distance between two points of  $\mathcal{M}_{j,D}$ . But for every  $m \neq m'$ ,

$$\begin{aligned} K(\mathbb{P}_{f_m}, \mathbb{P}_{f_{m'}}) &= n \int_{\mathbb{R}} f_{m'} \left( \exp \left( \log \frac{f_m}{f_{m'}} \right) - \log \frac{f_m}{f_{m'}} - 1 \right) \\ &= n \int_{\mathbb{R}} \left( f_m - f_{m'} - f_{m'} \log \left( 1 + \frac{f_m - f_{m'}}{f_{m'}} \right) \right) \\ &\leq n \int_{\mathbb{R}} \frac{(f_m - f_{m'})^2}{f_m} \\ &\leq \frac{2}{\rho} n \|f_m - f_{m'}\|_2^2 \\ &\leq \frac{2}{\rho} n D a_{j,D}^2, \end{aligned}$$

since  $\log(1+x) \geq x/(1+x)$ . So finally, following similar arguments to those used by [31] (pages 148 and 149), Fano's lemma implies that there exists an absolute constant  $c < 1$  such that

$$\mathcal{R}(\mathcal{M}_{j,D}) \geq \frac{(1-c)\theta'}{4} D a_{j,D}^2$$

as soon as the mean Kullback Leibler distance is small enough, which is implied by

$$\frac{2}{\rho} n D a_{j,D}^2 \leq c \sigma D \log(2^j/D). \quad (7.19)$$

Let us take  $j$  such that  $2^j \leq n/\log n \leq 2^{j+1}$  and with  $D \leq 2^j$ ,

$$a_{j,D}^2 = \frac{\rho^2}{4n} \log(2^j/D).$$

First note that (7.19) is automatically fulfilled as soon as  $\rho \leq 2c\sigma$ , that is true if  $\rho$  an absolute constant small enough. Then

$$\rho + 2^{j/2} |a_{j,D}| \leq \rho + 2^{j/2} \sqrt{\frac{\rho^2 \log n}{4n}} \leq 1.5\rho.$$

So, if  $\rho$  is an absolute constant small enough, (7.15) is satisfied. Moreover

$$2^{1+j/2}|a_{j,D}| \leq 2^{1+j/2} \sqrt{\frac{\rho^2 \log n}{4n}} \leq \rho.$$

This gives (7.14). Now, take an integer  $D = D_n$  such that

$$D_n \sim_{n \rightarrow \infty} R^{2/(1+2\alpha)} \left( \frac{n}{\log n} \right)^{1/(1+2\alpha)}.$$

For  $n$  large enough,  $D_n \leq 2^j$  and  $D_n$  is feasible. We have for  $R$  fixed,

$$a_{j,D_n}^2 \sim_{n \rightarrow \infty} c_\alpha \rho^2 \frac{\log n}{n},$$

where  $c_\alpha$  is a constant only depending on  $\alpha$ . Therefore,

$$\rho + 2^{j\alpha/(1+2\alpha)} \sqrt{D_n} |a_{j,D_n}| = \rho + \sqrt{c_\alpha} \rho R^{1/(1+2\alpha)} + o_n(1).$$

Since  $R^{1/(1+2\alpha)} \leq R'$  it is sufficient to take  $\rho$  small enough but constant depending only on  $\alpha$  to obtain (7.16). Moreover,

$$D_n a_{j,D_n}^2 \sim_{n \rightarrow \infty} c_\alpha \rho^2 R^{2/(1+2\alpha)} \left( \frac{\log n}{n} \right)^{2\alpha/(1+2\alpha)}.$$

Hence (7.17) is equivalent to  $\rho^2 < R^{2/(1+2\alpha)} \rho^{2\alpha/(1+2\alpha)}$ . Since  $R \geq 1$ , this is true as soon as  $\rho < 1$ . Finally (7.18) is equivalent, when  $n$  tends to  $+\infty$ , to

$$c_\alpha \rho^2 \leq (c_\alpha \rho)^{2\alpha/(1+2\alpha)}.$$

Once again this is true for  $\rho$  small enough depending on  $\alpha$ . As we can choose  $\rho$  not depending on  $R, R', R''$ , this concludes the proof.

Corollary 2 is completely straight forward once we notice that if  $R' \geq R$  then for every  $\alpha$ ,  $R' \geq R^{\frac{1}{1+2\alpha}}$ .

## 7.9 Proof of Theorem 8

Let  $\alpha > 1$  and  $n$  be fixed. We set  $j$  a positive integer such that

$$\frac{n}{(\log n)^\alpha} \leq 2^j < \frac{2n}{(\log n)^\alpha}.$$

For all  $k \in \{0, \dots, 2^j - 1\}$ , we define

$$N_{j,k}^+ = \int_{k2^{-j}}^{(k+1/2)2^{-j}} dN, \quad N_{j,k}^- = \int_{(k+1/2)2^{-j}}^{(k+1)2^{-j}} dN.$$

All these variables are iid random Poisson variables of parameter  $\mu_{n,j} = n2^{-j-1}$ . Moreover,

$$\hat{\beta}_{j,k} = \frac{2^{j/2}}{n} (N_{j,k}^+ - N_{j,k}^-) \text{ and } \hat{V}_{(j,k),n} = \frac{2^j}{n^2} (N_{j,k}^+ + N_{j,k}^-).$$

Hence,

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \geq \sum_{k=0}^{2^j-1} \frac{2^j}{n^2} \mathbb{E} \left( (N_{j,k}^+ - N_{j,k}^-)^2 1_{|N_{j,k}^+ - N_{j,k}^-| \geq \sqrt{2\gamma \log(n)(N_{j,k}^+ + N_{j,k}^-) + \log(n)u_n}} \right).$$

Denote by  $v_{n,j} = (\sqrt{4\gamma \log(n)\mu_{n,j}} + \log(n)u_n)^2$ . Remark that if  $N_{j,k}^+ = \mu_{n,j} + \frac{\sqrt{v_{n,j}}}{2}$  and  $N_{j,k}^- = \mu_{n,j} - \frac{\sqrt{v_{n,j}}}{2}$ , then

$$|N_{j,k}^+ - N_{j,k}^-| = \sqrt{2\gamma \log(n)(N_{j,k}^+ + N_{j,k}^-) + \log(n)u_n}.$$

Let  $N^+$  and  $N^-$  be two independent Poisson variables of parameter  $\mu_{n,j}$ . Then,

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \geq \frac{2^j}{n^2} v_{n,j} \mathbb{P} \left( N^+ = \mu_{n,j} + \frac{\sqrt{v_{n,j}}}{2} \text{ and } N^- = \mu_{n,j} - \frac{\sqrt{v_{n,j}}}{2} \right).$$

Note that

$$\frac{1}{4}(\log n)^\alpha < \mu_{n,j} \leq \frac{1}{2}(\log n)^\alpha,$$

and

$$\lim_{n \rightarrow +\infty} \frac{\sqrt{v_{n,j}}}{\mu_{n,j}} = 0.$$

So,  $l_{n,j} = \mu_{n,j} + \frac{\sqrt{v_{n,j}}}{2}$  and  $m_{n,j} = \mu_{n,j} - \frac{\sqrt{v_{n,j}}}{2}$  go to  $+\infty$  with  $n$ . Hence by Stirling formula,

$$\begin{aligned} \mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) &\geq \frac{v_{n,j}}{(\log n)^{2\alpha}} \mathbb{P} \left( N^+ = \mu_{n,j} + \frac{\sqrt{v_{n,j}}}{2} \right) \mathbb{P} \left( N^- = \mu_{n,j} - \frac{\sqrt{v_{n,j}}}{2} \right) \\ &\geq \frac{v_{n,j}}{(\log n)^{2\alpha}} \frac{\mu_{n,j}^{l_{n,j}}}{l_{n,j}!} e^{-\mu_{n,j}} \frac{\mu_{n,j}^{m_{n,j}}}{m_{n,j}!} e^{-\mu_{n,j}} \\ &\geq \frac{4\gamma \mu_{n,j}}{(\log n)^{2\alpha-1}} \left( \frac{\mu_{n,j}}{l_{n,j}} \right)^{l_{n,j}} e^{-(\mu_{n,j}-l_{n,j})} \left( \frac{\mu_{n,j}}{m_{n,j}} \right)^{m_{n,j}} e^{-(\mu_{n,j}-m_{n,j})} \frac{(1+o_n(1))}{2\pi \sqrt{l_{n,j}m_{n,j}}} \\ &\geq \frac{2\gamma}{\pi (\log n)^{2\alpha-1}} e^{-\mu_{n,j}} \left[ h\left(\frac{\sqrt{v_{n,j}}}{2\mu_{n,j}}\right) + h\left(-\frac{\sqrt{v_{n,j}}}{2\mu_{n,j}}\right) \right] (1+o_n(1)) \end{aligned}$$

where  $h(x) = (1+x)\log(1+x) - x = x^2/2 + O(x^3)$ . So,

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \geq \frac{2\gamma}{\pi (\log n)^{2\alpha-1}} e^{-\frac{v_{n,j}}{4\mu_{n,j}} + O_n\left(\frac{v_{n,j}^{3/2}}{\mu_{n,j}^2}\right)} (1+o_n(1)).$$

Since

$$v_{n,j} = 4\gamma \log(n)\mu_{n,j}(1+o_n(1)),$$

we obtain

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \geq \frac{2\gamma}{\pi (\log n)^{2\alpha-1}} e^{-\gamma \log(n) + o_n(\log(n))} (1+o_n(1)).$$

Finally, for every  $\varepsilon > 0$ ,

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \geq \frac{1}{n^{\gamma+\varepsilon}} (1+o_n(1)).$$

### 7.10 Proof of Theorem 9

We use notations of Lemma 4. Let  $f \in \mathcal{F}_n$ . We apply (7.6) with  $\varepsilon = 1.4$ . Then, with  $\gamma = 1 + \sqrt{2}$ , and  $\delta > 0$  such that  $(1 + \delta)^2 = 11.8/(2\gamma \times (1 + 2/\varepsilon)) \simeq 1.006$ , (7.6) becomes

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_\varphi^2) \leq \inf_{m \subset \Gamma_n} \left\{ 6 \sum_{\lambda \notin m} \beta_\lambda^2 + \sum_{\lambda \in m} [3.4 + 11.8 \log n] V_{\lambda,n} + c(\delta, \gamma)(1 + 2\varepsilon^{-1}) \sum_{\lambda \in m} \left( \frac{\log n \|\varphi_\lambda\|_\infty}{n} \right)^2 \right\} + \frac{C_2(\gamma, \|f\|_1, c, c', \varphi)}{n}.$$

Now, take

$$m = \{\lambda \in \Gamma_n : \beta_\lambda^2 > V_{\lambda,n}\}.$$

If  $m$  is empty, then  $\beta_\lambda^2 = \min(\beta_\lambda^2, V_{\lambda,n})$  for every  $\lambda$  of  $\Gamma_n$ . Hence

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_\varphi^2) \leq 6 \sum_{\lambda \in \Gamma_n} \beta_\lambda^2 + \frac{C_2(\gamma, \|f\|_1, c, c', \varphi)}{n}.$$

The result is true for  $n$  large enough even if the  $\beta_\lambda$ 's are all zero and this explains the presence of  $1/n$  in the oracle ratio.

If  $m$  is not empty, note  $\lambda = (j, k)$ . Since  $F_\lambda \leq 2^{-j} \|f\|_\infty$ , if  $F_\lambda \neq 0$ , then  $2^j = O(n/\log n)$  and  $\lambda \in \Gamma_n$ . Since

$$|\beta_\lambda| \leq S_\varphi 2^{j/2} F_\lambda,$$

this implies that  $F_\lambda$  is non zero for all  $\lambda \in m$ , and that if  $\beta_\lambda \neq 0$  then  $\lambda \in \Gamma_n$ . Now,

$$V_{\lambda,n} = \frac{1}{n} \sigma_\lambda^2 \geq \frac{1}{n} 2^j I_\varphi^2 F_\lambda \geq \frac{1}{n \Theta_\varphi} \|\varphi_\lambda\|_\infty^2 F_\lambda.$$

Hence, for all  $n$ , if  $\lambda \in m$ ,

$$V_{\lambda,n} \log n \geq \frac{(\log n)^2 (\log \log n)}{\Theta_\varphi n^2} \|\varphi_\lambda\|_\infty^2$$

and if  $n$  is large enough,

$$0.2 \log n \sum_{\lambda \in m} V_{\lambda,n} \geq c(\delta, \gamma)(1 + 2\varepsilon^{-1}) \sum_{\lambda \in m} \left( \frac{\log n}{n} \right)^2 \|\varphi_\lambda\|_\infty^2 + 3.4 \sum_{\lambda \in m} V_{\lambda,n}.$$

### 7.11 Proof of Theorem 10

Before proving Theorem 10, let us state the following result.

**Proposition 2.** *Let  $\gamma_{\min} \in (1, \gamma)$  be fixed and let  $\eta_{\lambda, \gamma_{\min}}$  be the threshold associated with  $\gamma_{\min}$ :*

$$\eta_{\lambda, \gamma_{\min}} = \sqrt{2\gamma_{\min} \log n \tilde{V}_{\lambda,n}} + \frac{\gamma_{\min} \log n}{3n} \|\varphi_\lambda\|_\infty,$$

where

$$\tilde{V}_{\lambda,n} = \hat{V}_{\lambda,n} + \sqrt{2\gamma_{\min} \log n \hat{V}_{\lambda,n} \frac{\|\varphi_\lambda\|_\infty^2}{n^2}} + 3\gamma_{\min} \log n \frac{\|\varphi_\lambda\|_\infty^2}{n^2}$$

(see Theorem 1). Let  $u = (u_n)_n$  be some sequence of positive numbers and

$$\Lambda_u = \{\lambda \text{ such that } \mathbb{P}(\eta_{\lambda,\gamma} > |\beta_\lambda| + \eta_{\lambda,\gamma_{\min}}) \geq 1 - u_n\}.$$

Then

$$\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) \geq \left( \sum_{\lambda \in \Lambda_u} \beta_\lambda^2 \right) (1 - (3n^{-\gamma_{\min}} + u_n)).$$

**Proof.**

$$\begin{aligned} \mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) &\geq \sum_{\lambda \in \Lambda_u} \mathbb{E} \left( (\hat{\beta}_\lambda - \beta_\lambda)^2 1_{|\hat{\beta}_\lambda| \geq \eta_{\lambda,\gamma}} + \beta_\lambda^2 1_{|\hat{\beta}_\lambda| < \eta_{\lambda,\gamma}} \right). \\ &\geq \sum_{\lambda \in \Lambda_u} \beta_\lambda^2 \mathbb{P}(|\hat{\beta}_\lambda| < \eta_{\lambda,\gamma}) \\ &\geq \sum_{\lambda \in \Lambda_u} \beta_\lambda^2 \mathbb{P}(|\hat{\beta}_\lambda - \beta_\lambda| + |\beta_\lambda| < \eta_{\lambda,\gamma}) \\ &\geq \sum_{\lambda \in \Lambda_u} \beta_\lambda^2 \mathbb{P}(|\hat{\beta}_\lambda - \beta_\lambda| < \eta_{\lambda,\gamma_{\min}} \text{ and } \eta_{\lambda,\gamma_{\min}} + |\beta_\lambda| < \eta_{\lambda,\gamma}) \\ &\geq \sum_{\lambda \in \Lambda_u} \beta_\lambda^2 \left( 1 - \left( \mathbb{P}(|\hat{\beta}_\lambda - \beta_\lambda| \geq \eta_{\lambda,\gamma_{\min}}) + \mathbb{P}(\eta_{\lambda,\gamma_{\min}} + |\beta_\lambda| \geq \eta_{\lambda,\gamma}) \right) \right) \\ &\geq \left( \sum_{\lambda \in \Lambda_u} \beta_\lambda^2 \right) (1 - (3n^{-\gamma_{\min}} + u_n)), \end{aligned}$$

by applying Lemma 1. ■

Using this proposition, we give the proof of Theorem 10. Let us consider

$$f = 1_{[0,1]} + \sum_{k \in \mathcal{N}_j} \sqrt{\frac{2(\sqrt{\gamma} - \sqrt{\gamma_{\min}})^2 \log n}{n}} \tilde{\varphi}_{j,k},$$

with

$$\mathcal{N}_j = \{0, 1, \dots, 2^j - 1\}$$

and

$$\frac{n}{(\log n)^{1+\alpha}} < 2^j \leq \frac{2n}{(\log n)^{1+\alpha}}, \quad \alpha > 0.$$

Note that for any  $(j, k)$ , if  $F_{j,k} \neq 0$ , then  $F_{j,k} = 2^{-j} \geq \frac{(\log n)(\log \log n)}{n}$  for  $n$  large enough and  $f$  belongs to  $\mathcal{F}_n$ . Furthermore,  $V_{(-1,0),n} = \frac{1}{n}$  and for any  $k \in \mathcal{N}_j$ ,  $V_{(j,k),n} = \frac{1}{n}$ . So, for  $n$  large enough,

$$\sum_{\lambda \in \Gamma_n} \min(\beta_\lambda^2, V_{\lambda,n}) = V_{(-1,0),n} + \sum_{k \in \mathcal{N}_j} V_{(j,k),n} = \frac{1}{n} + \sum_{k \in \mathcal{N}_j} \frac{1}{n}.$$

Now, to apply Proposition 2, let us set for any  $n$ ,  $u_n = n^{-\gamma}$  and observe that for any  $\varepsilon > 0$ ,

$$\mathbb{P}(\eta_{\lambda,\gamma_{\min}} + |\beta_\lambda| \geq \eta_{\lambda,\gamma}) \leq \mathbb{P}((1 + \varepsilon)2\gamma_{\min} \log n \tilde{V}_{\lambda,n}(\gamma_{\min}) + (1 + \varepsilon^{-1})\beta_\lambda^2 > 2\gamma \log n \tilde{V}_{\lambda,n}(\gamma)),$$

since  $\gamma_{\min} < \gamma$ . With  $\varepsilon = \sqrt{\gamma/\gamma_{\min}} - 1$  and  $\theta = \sqrt{\gamma_{\min}/\gamma}$ ,

$$\begin{aligned} \mathbb{P}((1 + \varepsilon)2\gamma_{\min} \log n \tilde{V}_{\lambda,n}(\gamma_{\min}) + (1 + \varepsilon^{-1})\beta_\lambda^2 > 2\gamma \log n \tilde{V}_{\lambda,n}(\gamma)) = \\ \mathbb{P}(\theta \tilde{V}_{\lambda,n}(\gamma_{\min}) + (1 - \theta)V_{\lambda,n} > \tilde{V}_{\lambda,n}(\gamma)). \end{aligned}$$

Since  $\tilde{V}_{\lambda,n}(\gamma_{\min}) < \tilde{V}_{\lambda,n}(\gamma)$ ,

$$\mathbb{P}(\eta_{\lambda,\gamma_{\min}} + |\beta_{\lambda}| \geq \eta_{\lambda,\gamma}) \leq \mathbb{P}(V_{\lambda,n} > \tilde{V}_{\lambda,n}(\gamma)) \leq u_n.$$

So,

$$\{(j, k) : k \in \mathcal{N}_j\} \subset \Lambda_u,$$

and

$$\begin{aligned} \mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2) &\geq \sum_{k \in \mathcal{N}_j} \beta_{j,k}^2 (1 - (3n^{-\gamma_{\min}} + n^{-\gamma})) \\ &\geq (\sqrt{\gamma} - \sqrt{\gamma_{\min}})^2 2 \log n \sum_{k \in \mathcal{N}_j} \frac{1}{n} (1 - (3n^{-\gamma_{\min}} + n^{-\gamma})) \\ &\geq (\sqrt{\gamma} - \sqrt{\gamma_{\min}})^2 2 \log n \left( \sum_{\lambda \in \Gamma_n} \min(\beta_{\lambda}^2, V_{\lambda,n}) - \frac{1}{n} \right) (1 - (3n^{-\gamma_{\min}} + n^{-\gamma})). \end{aligned}$$

Finally, since  $\text{card}(\mathcal{N}_j) \rightarrow +\infty$  when  $n \rightarrow +\infty$ ,

$$\frac{\mathbb{E}(\|\tilde{f}_{n,\gamma} - f\|_{\tilde{\varphi}}^2)}{\sum_{\lambda \in \Gamma_n} \min(\beta_{\lambda}^2, V_{\lambda,n}) + \frac{1}{n}} \geq (\sqrt{\gamma} - \sqrt{\gamma_{\min}})^2 2 \log n (1 + o_n(1)).$$

■

## Appendix

The following table gives the definition of the signals used in Section 6.

Haar1  $\mathbf{1}_{[0,1]}$	Haar2  $1.5 \mathbf{1}_{[0,0.125]} + 0.5 \mathbf{1}_{[0.125,0.25]} + \mathbf{1}_{[0.25,1]}$	Blocks  $\left(2 + \sum_j \frac{h_j}{2} (1 + \text{sgn}(x - p_j))\right) \frac{\mathbf{1}_{[0,1]}}{3.551}$
Comb  $32 \sum_{k=1}^{+\infty} \frac{1}{k 2^k} \mathbf{1}_{[k^2/32, (k^2+k)/32]}$	Gauss1  $\frac{1}{0.25\sqrt{2\pi}} \exp\left(\frac{(x-0.5)^2}{2 \times 0.25^2}\right)$	Gauss2  $\frac{1}{\sqrt{2\pi}} \exp\left(\frac{(x-0.5)^2}{2 \times 0.25^2}\right) + \frac{3}{\sqrt{2\pi}} \exp\left(\frac{(x-5)^2}{2 \times 0.25^2}\right)$
Beta0.5  $0.5x^{-0.5} \mathbf{1}_{[0,1]}$	Beta4  $3x^4 \mathbf{1}_{[1,+\infty[}$	Bumps  $\left(\sum_j g_j \left(1 + \frac{ x - p_j }{w_j}\right)^{-4}\right) \frac{\mathbf{1}_{[0,1]}}{0.284}$

where

$$\begin{array}{lcl} \mathbf{p} & = & \begin{bmatrix} 0.1 & 0.13 & 0.15 & 0.23 & 0.25 & 0.4 & 0.44 & 0.65 & 0.76 & 0.78 & 0.81 & \end{bmatrix} \\ \mathbf{h} & = & \begin{bmatrix} 4 & -5 & 3 & -4 & 5 & -4.2 & 2.1 & 4.3 & -3.1 & 2.1 & -4.2 & \end{bmatrix} \\ \mathbf{g} & = & \begin{bmatrix} 4 & 5 & 3 & 4 & 5 & 4.2 & 2.1 & 4.3 & 3.1 & 5.1 & 4.2 & \end{bmatrix} \\ \mathbf{w} & = & \begin{bmatrix} 0.005 & 0.005 & 0.006 & 0.01 & 0.01 & 0.03 & 0.01 & 0.01 & 0.005 & 0.008 & 0.005 & \end{bmatrix} \end{array}$$



**Acknowledgment.** The authors acknowledge the support of the French Agence Nationale de la Recherche (ANR), under grant ATLAS (JCJC06\_137446) "From Applications to Theory in Learning and Adaptive Statistics". We would like to warmly thank Lucien Birgé for his advises and his encouragements.

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